Application of Tensor Renormalization Group to Particle Physics

CCS, Univ. of Tsukuba

Yoshinobu Kuramashi

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Plan of Talk

- Introduction
- Tensor Renormalization Group (TRG)
  - Two Dimensional Ising Model
- Current Status for TRG Studies in Particle Physics
- Application to Complex Actions
  - 2D pure U(1) Gauge Theory w/ θ-Term
- Summary

Introduction

What is Tensor Network (TN) Scheme?

Theoretical and numerical methods for high precision analyses of many body problems with tensor network formalism

What is different from conventional methods?

Free from sign problem and complex action problem in Monte Carlo method
Computational cost for L^D system size $\propto D \times \log(L)$
Direct treatment of Grassmann numbers
Direct evaluation of partition function Z itself

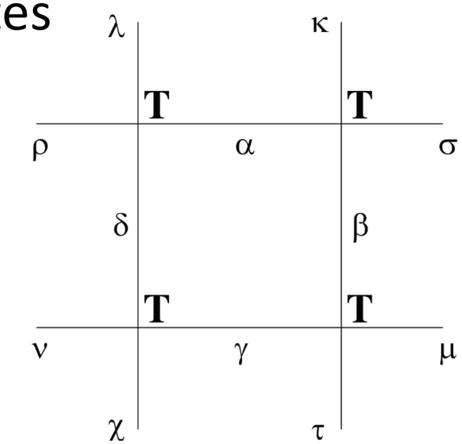
↓

Possible applications in particle physics :
Light quark dynamics in QED/QCD, Finite density QCD,
Strong CP problem, Chiral gauge theories, Lattice SUSY etc.
Also many applications in condensed matter physics

Tensor Renormalization Group (TRG)

Levin-Nave
PRL99(2007)120601

Explain the algorithm with 2D Ising model with N sites



Hamiltonian $H = \sum_{\langle i,j\rangle} s_i s_j \qquad s_i \pm 1$

Partition Function $Z = \sum_{\{s_i\}} \exp\left(-\beta H\right)$

$$= \sum_{\alpha,\beta,\gamma,\delta,\cdots=1}^{2} T_{\alpha,\lambda,\rho,\delta} T_{\sigma,\kappa,\alpha,\beta} T_{\mu,\beta,\gamma,\tau} T_{\gamma,\delta,\nu,\chi} \cdots$$

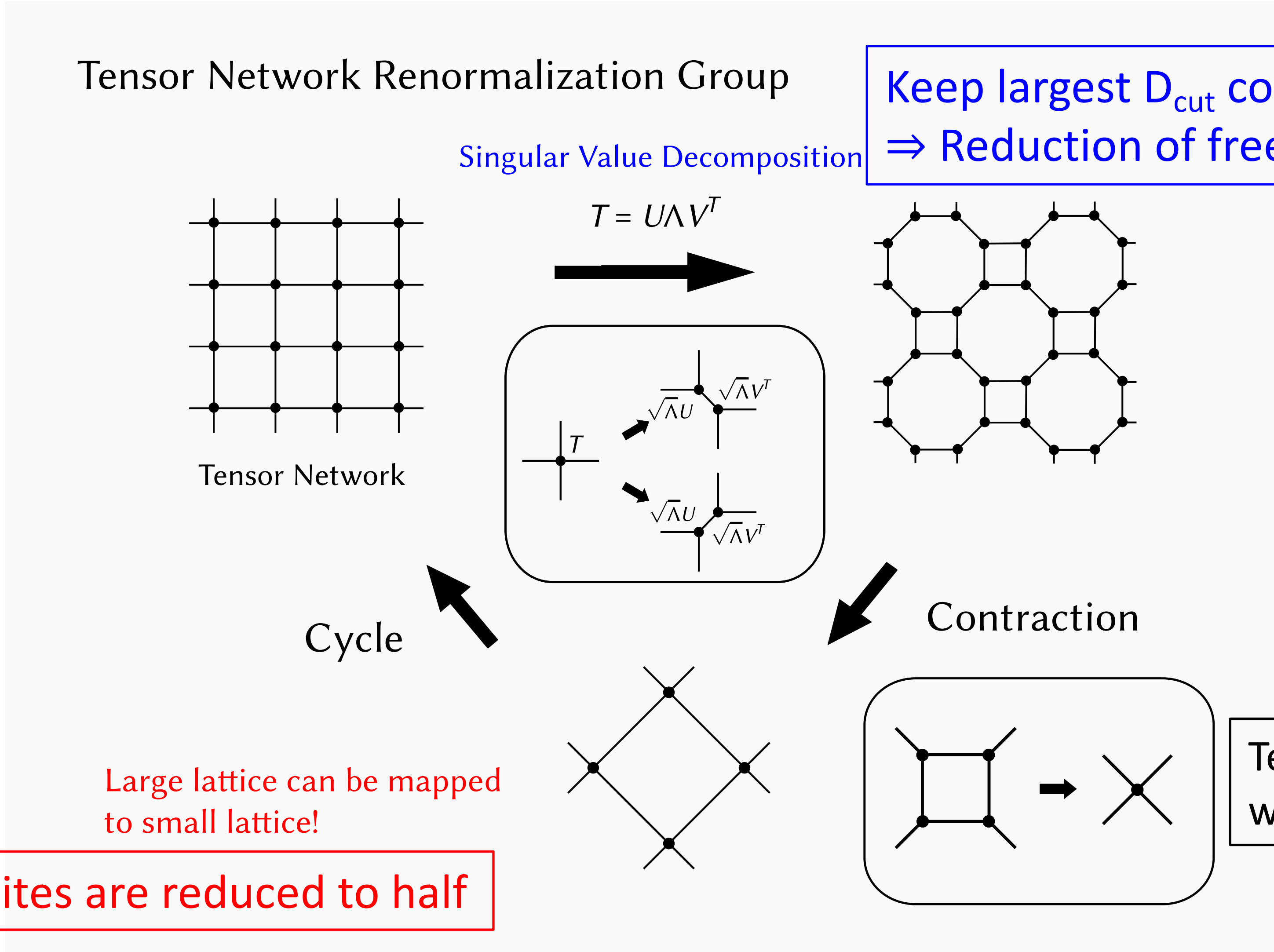
Tensor Network formulation

Details of model are specified in initial tensor
The algorithmic procedure is independent of models

Of course, direct contraction is impossible for large N even with current fastest supercomputer
⇒ How to evaluate the partition function?

Schematic View of TRG Algorithm

1. Singular Value Decomposition of local tensor T
2. Contraction of old tensor indices (coarse-graining)
3. Repeat the iteration



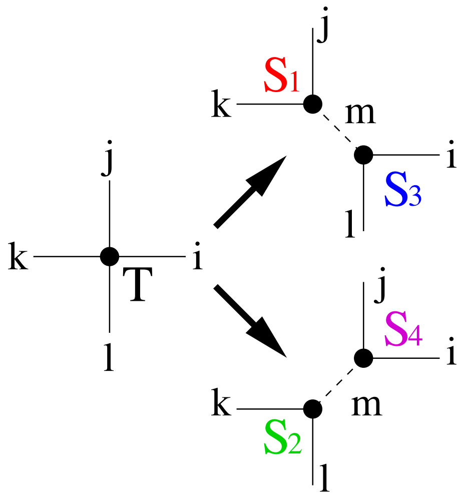
Keep largest D_{cut} components
⇒ Reduction of freedom

Tensors
w/ new indices

#sites are reduced to half

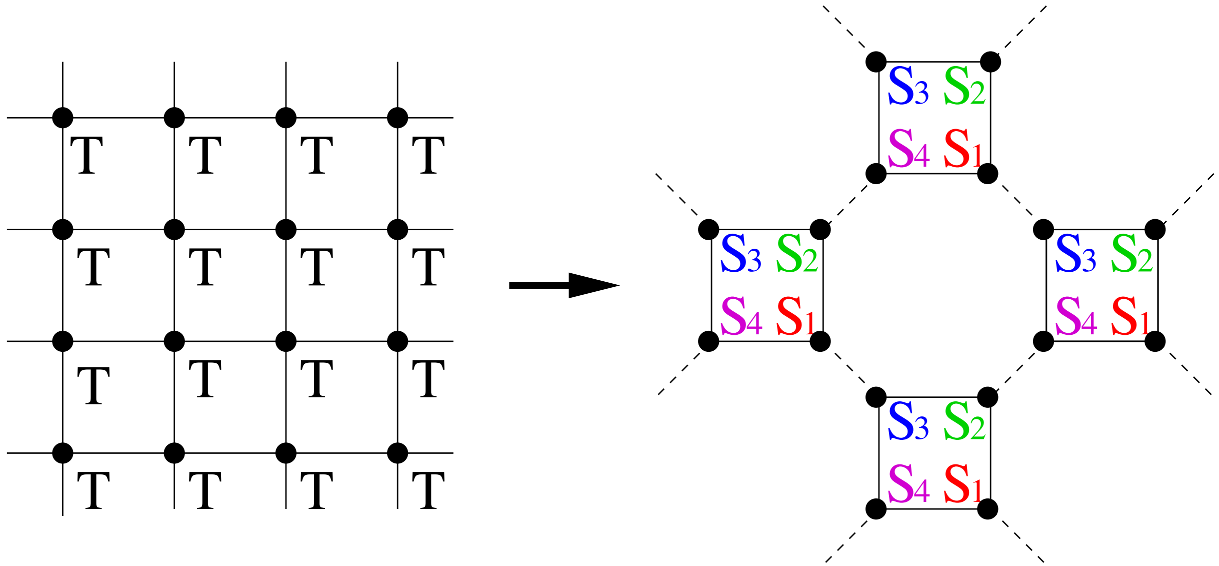
TRG Algorithm (1)

Singular Value Decomposition of local tensor T



$$T_{i,j,k,l} \Rightarrow T_{\{j,k\},\{l,i\}} = \left(U\Lambda V^t\right)_{\{j,k\},\{l,i\}} = \sum_m \left(U\sqrt{\Lambda}\right)_{\{j,k\},m} \left(V\sqrt{\Lambda}\right)_{\{l,i\},m} = \sum_m \left(S_1\right)_{\{j,k\},m} \left(S_3\right)_{\{l,i\},m}$$

$$T_{i,j,k,l} \Rightarrow T_{\{k,l\},\{i,j\}} = \left(U\Lambda V^t\right)_{\{k,l\},\{i,j\}} = \sum_m \left(U\sqrt{\Lambda}\right)_{\{k,l\},m} \left(V\sqrt{\Lambda}\right)_{\{i,j\},m} = \sum_m \left(S_2\right)_{\{k,l\},m} \left(S_4\right)_{\{i,j\},m}$$

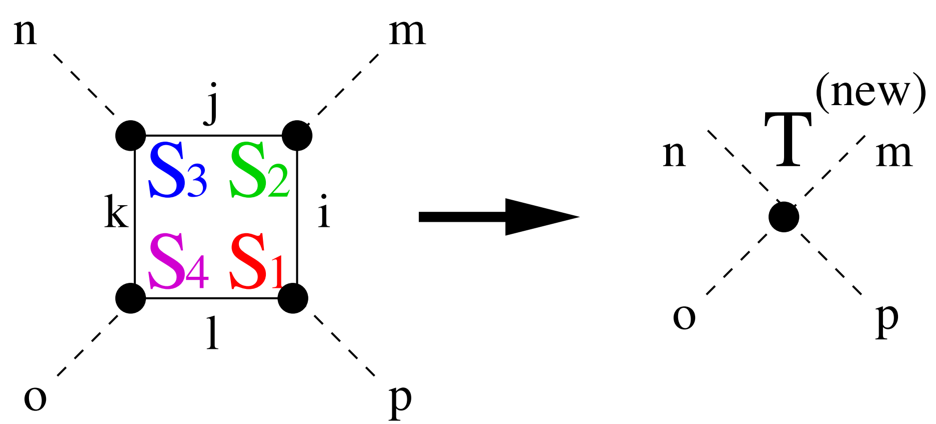


TRG Algorithm (2)

Keep largest D_{cut} components ⇒ Reduction of freedom

$$T_{i,j,k,l} \simeq \sum_{m=1}^{\mathrm{D_{cut}}} U_{\{k,l\},m} \Lambda_m V_{\{i,j\},m}$$

Contraction of old tensor indices of S_1, S_2, S_3, S_4 (coarse-graining)



$$T^{(\mathrm{new})}_{o,n,m,p} = \sum_{i,j,k,l} (S_4)_{\{l,k\},o} (S_3)_{\{k,j\},n} (S_2)_{\{j,i\},m} (S_1)_{\{i,l\},p}$$

#sites are reduced to half

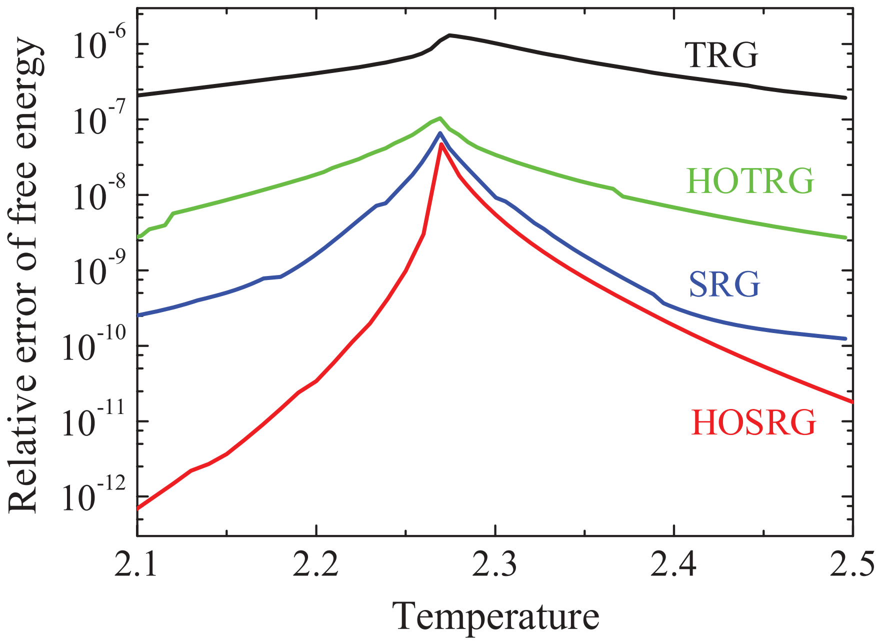
Numerical test for 2D Ising Model

The key element in the algorithm is low-rank approximation by SVD

$$T_{i,j,k,l} \simeq \sum_{m=1}^{\mathrm{D_{cut}}} U_{\{k,l\},m} \Lambda_m V_{\{i,j\},m}$$

Truncation error is controlled by the parameter $\mathrm{D_{cut}}$

Free energy on and off the transition point, lattice size=$2^{30\sim50}$, $\mathrm{D_{cut}}$=24



Xie et al.
PRB86(2012)045139

Comparison with analytic results

Relative error of free energy : $\leq 10^{-6}$

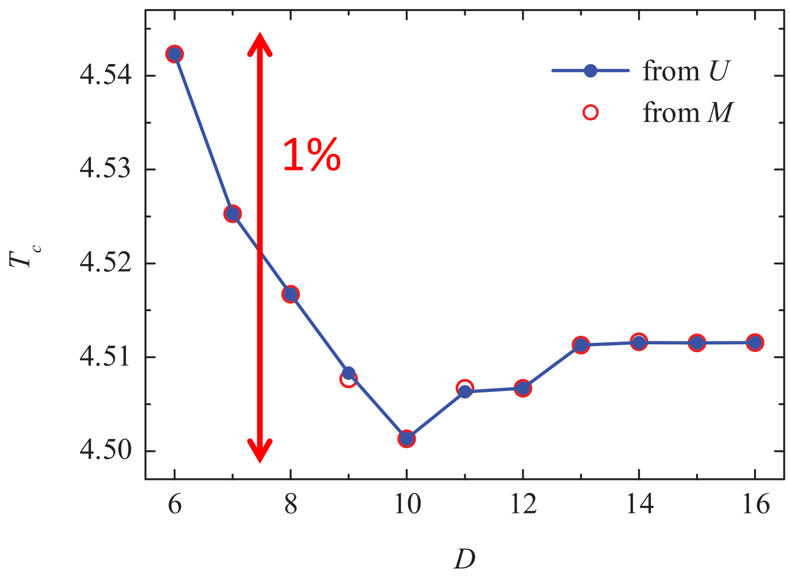
Study of 3D Ising Model

Xie et al.
PRB86(2012)045139

Higher-Order TRG (HOTRG): applicable to higher dimensional models

Computational cost $\propto (D_{cut})^{11} \times \log(V)$

D_{cut} dependence of T_c



Comparison with Monte Carlo data

Method	T_c
HOTRG ($D = 16$, from U)	4.511544
HOTRG ($D = 16$, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

Results show good agreement with Monte Carlo data at high precision

Collaborators

Y. Kuramashi, Y. Yoshimura S. Akiyama	U. Tsukuba
Y. Nakamura, (Y. Shimizu)	R-CCS
S. Takeda, R. Sakai	Kanazawa U.
D. Kadoh	N. Tsing-Hua U./ Keio U.

Application of TRGs to Particle Physics (1)

2D models

Ising model : Levin-Nave, PRL99(2007)120601

X-Y model : Meurice+, PRE89(2014)013308

CP(1) : Kawauchi-Takeda, PRD93(2016)114503

Real ϕ^4 theory :

Shimizu, Mod.Phys.Lett.A27(2012)1250035,

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP1905(2019)184

Complex ϕ^4 theory at finite density :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, in preparation

U(1) gauge theory+θ :

YK-Yoshimura, arXiv:1911.06480

Schwinger, Schwinger+θ :

Shimizu-YK, PRD90(2014)014508, PRD90(2014)074503, PRD97(2018)034502

Gross-Neveu model at finite density :

Takeda-Yoshimura, PTEP2015(2015)043B01

N=1 Wess-Zumino model :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP1803(2018)141

Application of TRGs to Particle Physics (2)

3D models

Ising : Xie+, PRB86(2012)045139

Potts model : Wan+, CPL31(2014)070503

Free Wilson fermion :

Sakai-Takeda-Yoshimura, PTEP2017(2017)063B07,

Yoshimura-YK-Nakamura-Takeda-Sakai, PRD97(2018)054511

Z_2 gauge theory at finite temperature :

YK-Yoshimura, JHEP1908(2019)023

4D models

Ising : Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510

weak first-order phase transition (not second-order phase transition)

A Selected Topic on Complex Action

2D pure U(1) gauge theory w/ θ-term
YK-Yoshimura, arXiv:1911.06480

How to treat continuous dof?
Complex action with so-called θ-term
Sign problem is really solved?

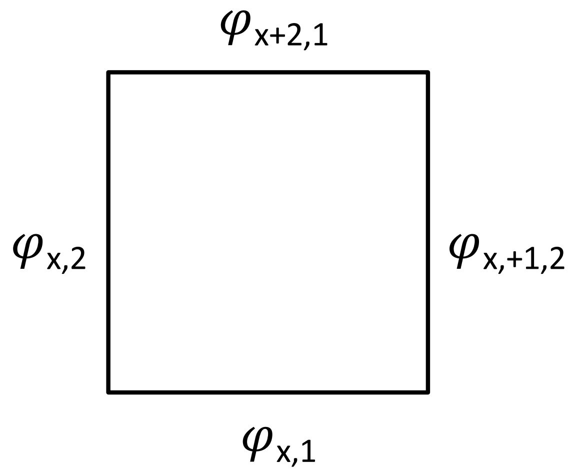
2D Pure U(1) Gauge Theory w/ θ-Term

Lattice action of 2D pure U(1) gauge theory w/ θ-term

$$S = -\beta \sum_x \cos p_x - i\theta Q \quad \text{complex}$$

$$p_x = \varphi_{x,1} + \varphi_{x+\hat{1},2} - \varphi_{x+\hat{2},1} - \varphi_{x,2}$$

$$Q = \frac{1}{2\pi}\sum_x q_x, \qquad q_x = p_x \bmod 2\pi$$



Periodic boundary condition ⇒ Q is integer

$$Z = \left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{d\varphi_{x,\mu}}{2\pi}\right)\exp\left(-S\right)$$

Analytic result

$$Z_{\text{analytic}} = \sum_{Q=-\infty}^{\infty}\left(z_{\text{P}}(\theta + 2\pi Q, \beta)\right)^V,$$

$$z_{\text{P}}(\theta, \beta) = \int_{-\pi}^{\pi}\frac{d\varphi_{\text{P}}}{2\pi}\exp\left(\beta\cos\varphi_{\text{P}} + i\frac{\theta}{2\pi}\varphi_{\text{P}}\right)$$

Wiese, NPB318(1989)153

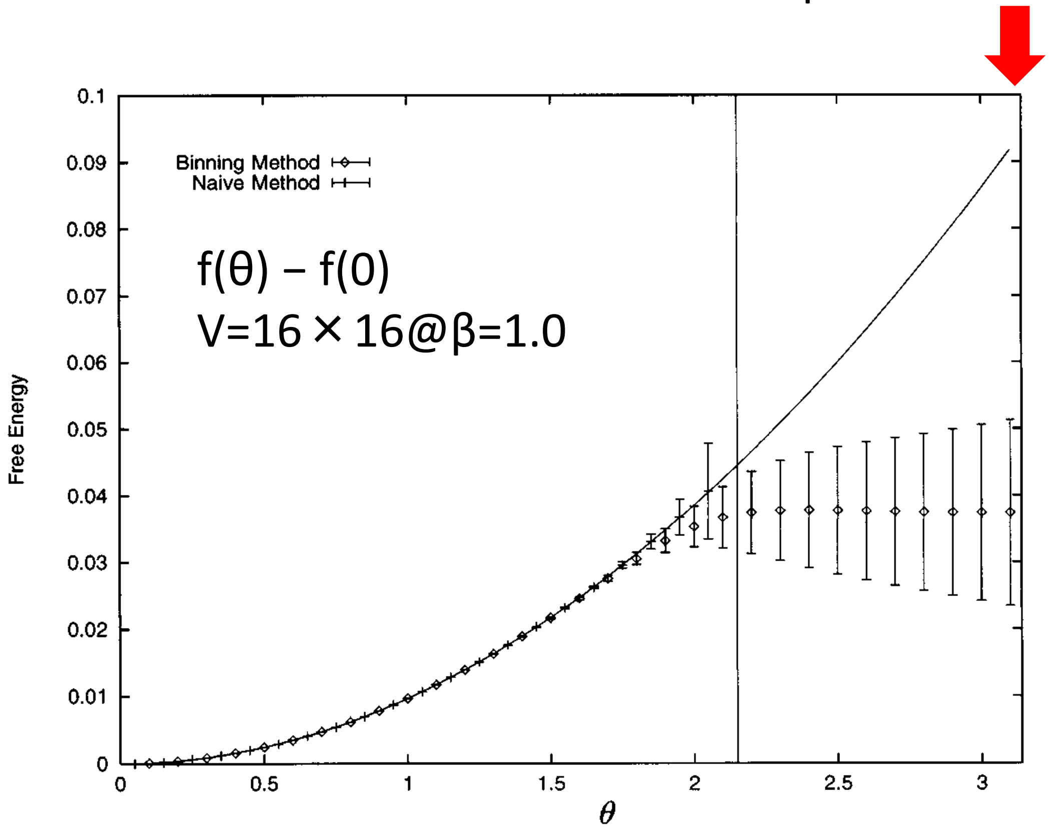
Predict a first order phase transition at θ=π

Conventional Approach

Plefka-Samuel, PRD56(1997)44

Previous Monte Carlo result for free energy

Expected transition point



Doesn’t work for $\theta \gtrsim 0.7\pi$ (difficult near the transition point)

No successful numerical calculation so far

Tensor Network Representation

YK-Yoshimura, arXiv:1911.06480

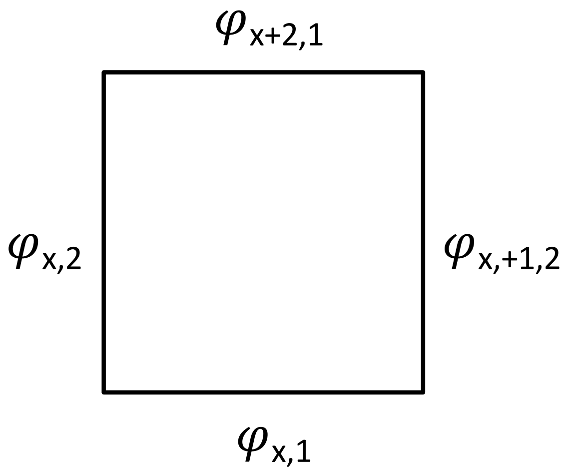
Lattice action of 2D pure U(1) gauge theory w/ θ-term

$$S=-\beta\sum_x \cos p_x - i\theta Q \quad \text{complex}$$

$$p_x=\varphi_{x,1}+\varphi_{x+\hat{1},2}-\varphi_{x+\hat{2},1}-\varphi_{x,2}$$

$$Q=\frac{1}{2\pi}\sum_x q_x,\quad q_x=p_x \bmod 2\pi$$

$$Z=\left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{d\varphi_{x,\mu}}{2\pi}\right)\exp(-S)$$



Partition function w/ continuous indices

$$\mathcal{T}(\varphi_{x,1},\varphi_{x+\hat{1},2},\varphi_{x+\hat{2},1},\varphi_{x,2})=\exp\left(\beta\cos p_x + i\frac{\theta}{2\pi}q_x\right)$$

$$Z=\left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{d\varphi_{x,\mu}}{2\pi}\right)\prod_x \mathcal{T}(\varphi_{x,1},\varphi_{x+\hat{1},2},\varphi_{x+\hat{2},1},\varphi_{x,2})$$

Discretized partition function w/ Gauss-Legendre quadrature

$$\int d\varphi f(\varphi)\approx\sum_{\alpha=1}^{K}w_\alpha f\left(\varphi^{(\alpha)}\right)$$

K-th order Legendre polynomial

$\varphi^{(\alpha)}$: α-th node, w_α : weight

$$T_{ijkl}=\frac{\sqrt{w_iw_jw_kw_l}}{(2\pi)^2}\mathcal{T}\left(\varphi^{(i)},\varphi^{(j)},\varphi^{(k)},\varphi^{(l)}\right)$$

$$Z\approx\sum_{\{\alpha\}}\prod_x T_{n_{x,1}n_{x+\hat{1},2}n_{x+\hat{2}}n_{x,2}}$$

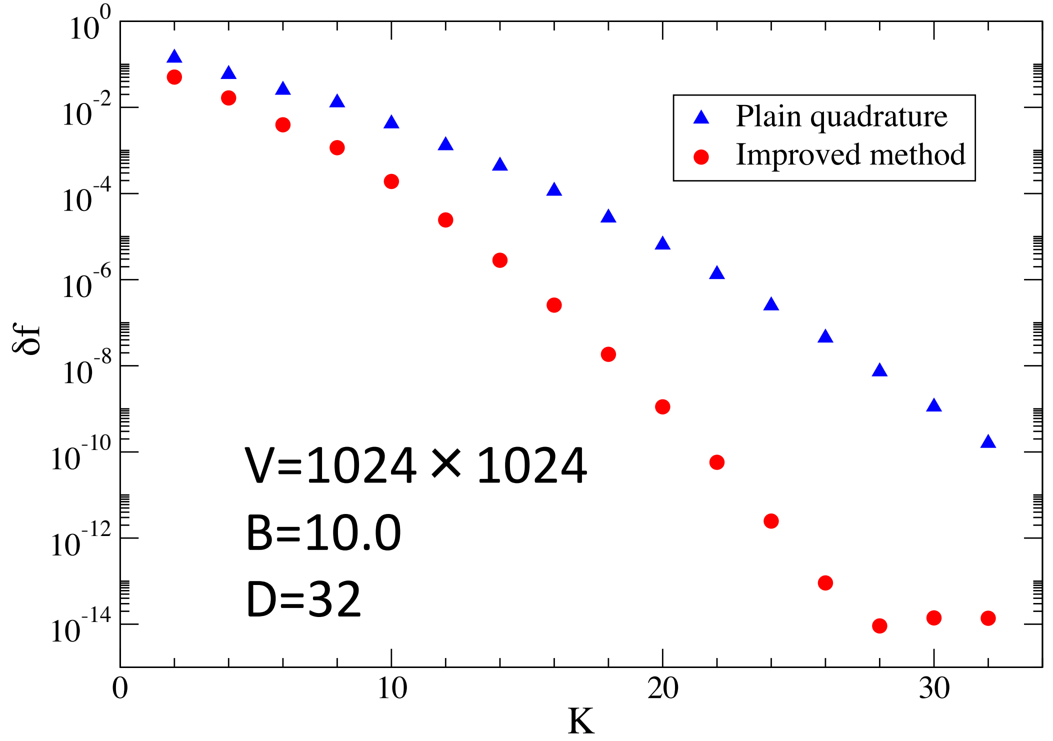
Free Energy w/ TRG

YK-Yoshimura, arXiv:1911.06480

Relative error

$$\delta f = \frac{|\ln Z_{\mathrm{analytic}} - \ln Z(K, D=32)|}{|\ln Z_{\mathrm{analytic}}|}$$

Comparison btw TRG results and analytic ones at θ=π (most difficult point)



$\delta f < 10^{-12}$@(D,K)=(32,32) ⇒ sufficient precision

Topological Charge Density

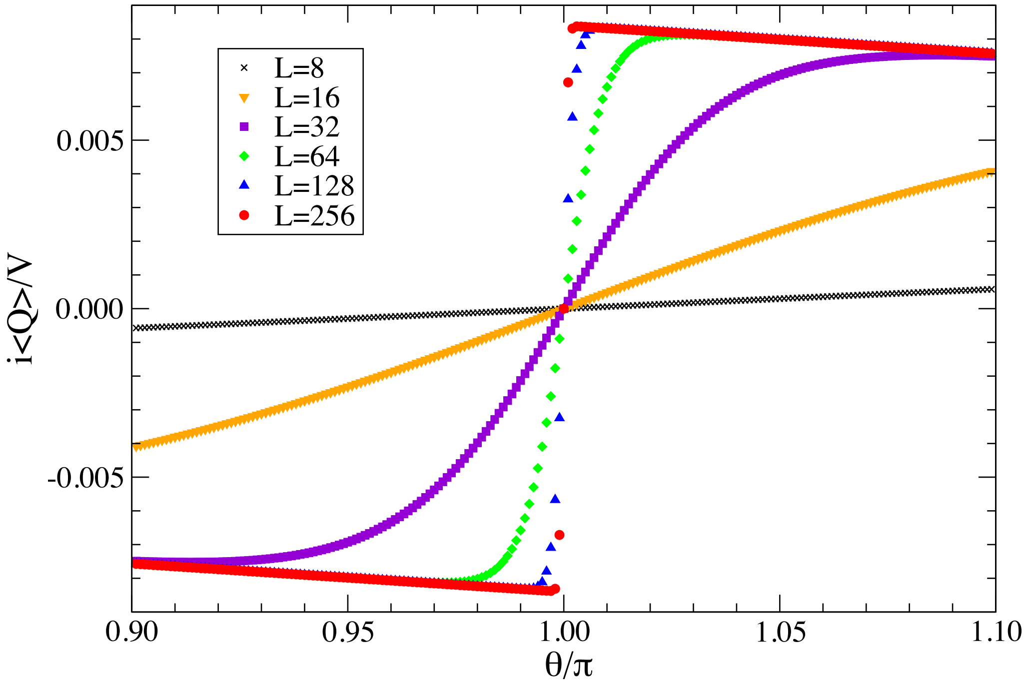
YK-Yoshimura, arXiv:1911.06480

Expectation value of topological charge (order parameter in this model)

$$\langle Q \rangle = -i \frac{\partial \ln Z}{\partial \theta}.$$

Numerical derivative in terms of θ

Volume dependence of topological charge density around θ=π



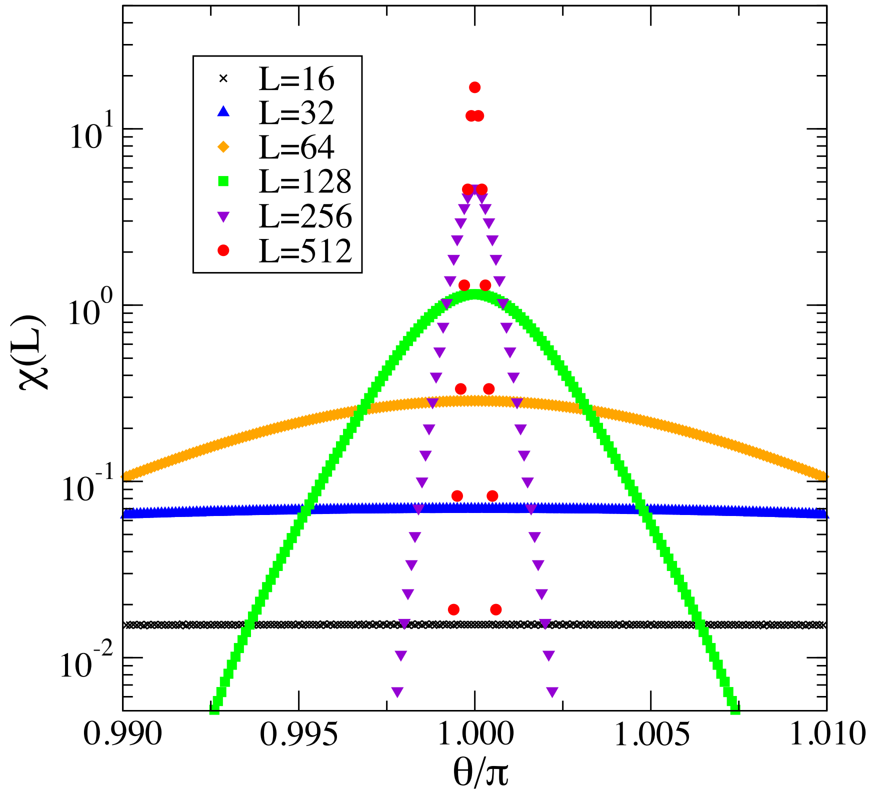
Finite gap is generated toward L→∞ ⇒ 1st order phase transition

Topological Susceptibility

YK-Yoshimura, arXiv:1911.06480

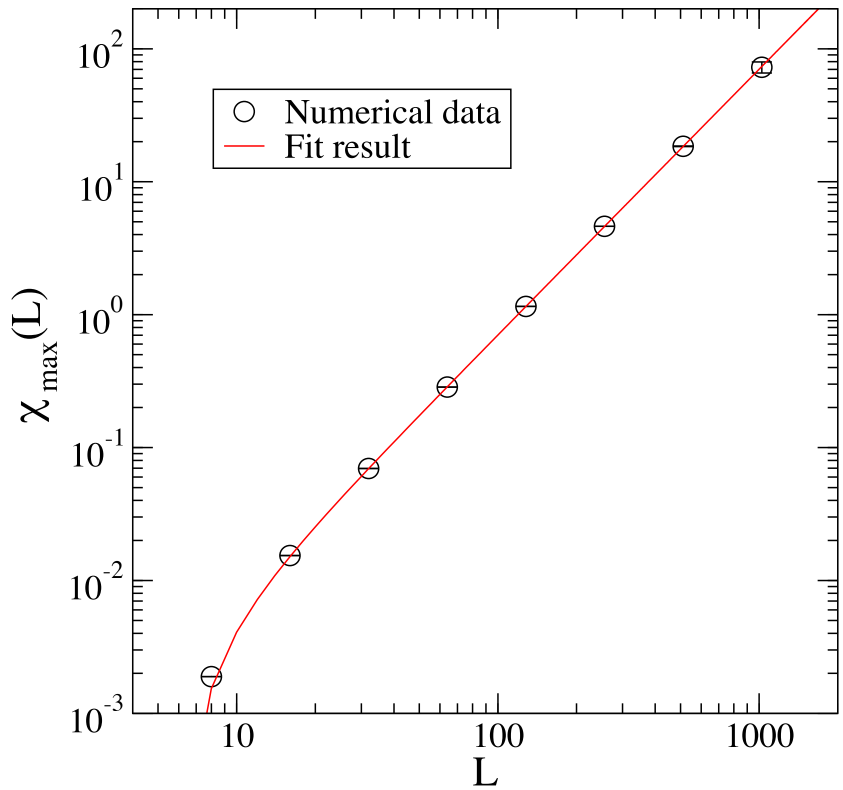
Topological susceptibility

$$\chi(L) = -\frac{1}{V}\frac{\partial^2 \ln Z}{\partial \theta^2}.$$



FSS of peak height

$$\chi_{\max}(L) = A + BL^{\gamma/\nu}$$



Critical exponent γ/ν=1.998(2) ⇒ 1st order phase transition

Summary

What we have achieved so far

- Application of TRG/HOTRG to 2D scalar, fermion and gauge theories
  - Construction of TN rep. for scalar field theories
  - Development of Grassmann TRG/HOTRG for fermion systems
  - Successful analyses of 2D models w/ complex actions

- Analyses of higher dimensional models
  - Phase transition of 3D Z_2 gauge theory
  - Phase transition of 4D Ising model

Next step

- Analysis of 3D, 4D models
  - Non-Abelian gauge theories