On the completeness of RPA solutions

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Contents :

I. Introduction

RPA — a standard method in quantum many-body theory

- **constructing eigenmodes in vicinity of MF state**
  $\to$ excited states, linear response, *etc.*
- **extensions $\cdots$ QRPA, rel.-RPA, 2nd RPA, extended RPA, *etc.***
- **small amp. limit of TDMF (& TDDFT)**

RPA eq. $\mathsf{S}\,\boldsymbol{x}_\nu = \omega_\nu \mathsf{N}\,\boldsymbol{x}_\nu$ $\qquad (\omega_\nu, \boldsymbol{x}_\nu)$ **: solution**

$$\mathsf{S} := \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \ (\text{with } A = A^\dagger, B = B^T)\,, \ \mathsf{N} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- **generalized eigenvalue problem = *e.g.* eigenvalue problem of $\mathsf{N}\,\mathsf{S}$**
  $\Rightarrow$ general properties of solutions — non-trivial
  completeness of solutions ? — questionable
- **normalization : $\boldsymbol{x}_\nu^\dagger \mathsf{N}\,\boldsymbol{x}_\nu = 1$ $\quad\Rightarrow\quad$ "normalizability" : $\boldsymbol{x}_\nu^\dagger \mathsf{N}\,\boldsymbol{x}_\nu > 0$**
  $\cdots$ not necessarily guaranteed, a key to solvability

Mathematical properties of RPA solutions

solutions at and in vicinity of stability
— studied by D.J. Thouless in '60s

Thouless, N.P.A 21, 225; 22, 78
Thouless & Valatin, N.P.A 31, 221

Nobel prize in physics, 2016
(from APS Journal website)

complete picture ? (↔ coding)

cf. SSB in MF solution (*e.g.* translation, rotation)
→ sitting near the borderline of stability
→ sometimes diving into instability due to numerical errors
or to additional approximations

II. Dualities

Properties of stability matrix S (*i.e.* RPA Hamiltonian)

$$\mathsf{S} = \mathsf{S}^\dagger\,, \quad \Sigma_x\,\mathsf{S}^*\,\Sigma_x = \mathsf{S}\,; \quad \Sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow \textbf{2 types of “dualities”}$$

★ **Dualities for RPA solutions**

- **UL-duality**

$$\mathsf{S}\,\boldsymbol{x}_\nu = \omega_\nu \mathsf{N}\,\boldsymbol{x}_\nu \quad \to \quad \mathsf{S}\,(\Sigma_x \boldsymbol{x}_\nu^*) = -\omega_\nu^*\,\mathsf{N}\,(\Sigma_x \boldsymbol{x}_\nu^*)$$

i.e. **solution** $(\omega_\nu, \boldsymbol{x}_\nu) \quad \to \quad (-\omega_\nu^*, \Sigma_x \boldsymbol{x}_\nu^*)$

- **LR-duality**

$$\begin{cases} \mathsf{N}\,\mathsf{S}\,\boldsymbol{x}_\nu = \omega_\nu\,\boldsymbol{x}_\nu & \textbf{(right eigenvalue problem of } \mathsf{N}\,\mathsf{S}\textbf{)} \\ \mathsf{S}\,\mathsf{N}\,(\mathsf{N}\boldsymbol{x}_\nu) = \omega_\nu\,(\mathsf{N}\boldsymbol{x}_\nu) & \\ \quad \to \quad (\boldsymbol{x}_\nu^\dagger \mathsf{N})\,\mathsf{N}\,\mathsf{S} = (\boldsymbol{x}_\nu^\dagger \mathsf{N})\,\omega_\nu^* & \textbf{(left eigenvalue problem of } \mathsf{N}\,\mathsf{S}\textbf{)} \end{cases}$$

i.e. **right eigensolution** $(\omega_\nu, \boldsymbol{x}_\nu) \quad \to \quad$ **left eigensolution** $(\omega_\nu^*, \boldsymbol{x}_\nu^\dagger \mathsf{N})$

★ Complete classification of RPA solutions

(i) $\omega_\nu > 0$ **with a normalizable eigenvector** $\boldsymbol{x}_\nu$**,**
in association with another eigensolution $(-\omega_\nu, \Sigma_x \boldsymbol{x}_\nu^*)$**.**

(ii) $\omega_\nu > 0$ **with an unnormalizable eigenvector, in association with an eigenvalue** $-\omega_\nu$ **that could be normalizable.**

(iii) **a pair of pure imaginary eigenvalues** $\pm\omega_\nu$
with $\mathrm{Re}(\omega_\nu) = 0$**,** $\mathrm{Im}(\omega_\nu) \neq 0$**.**

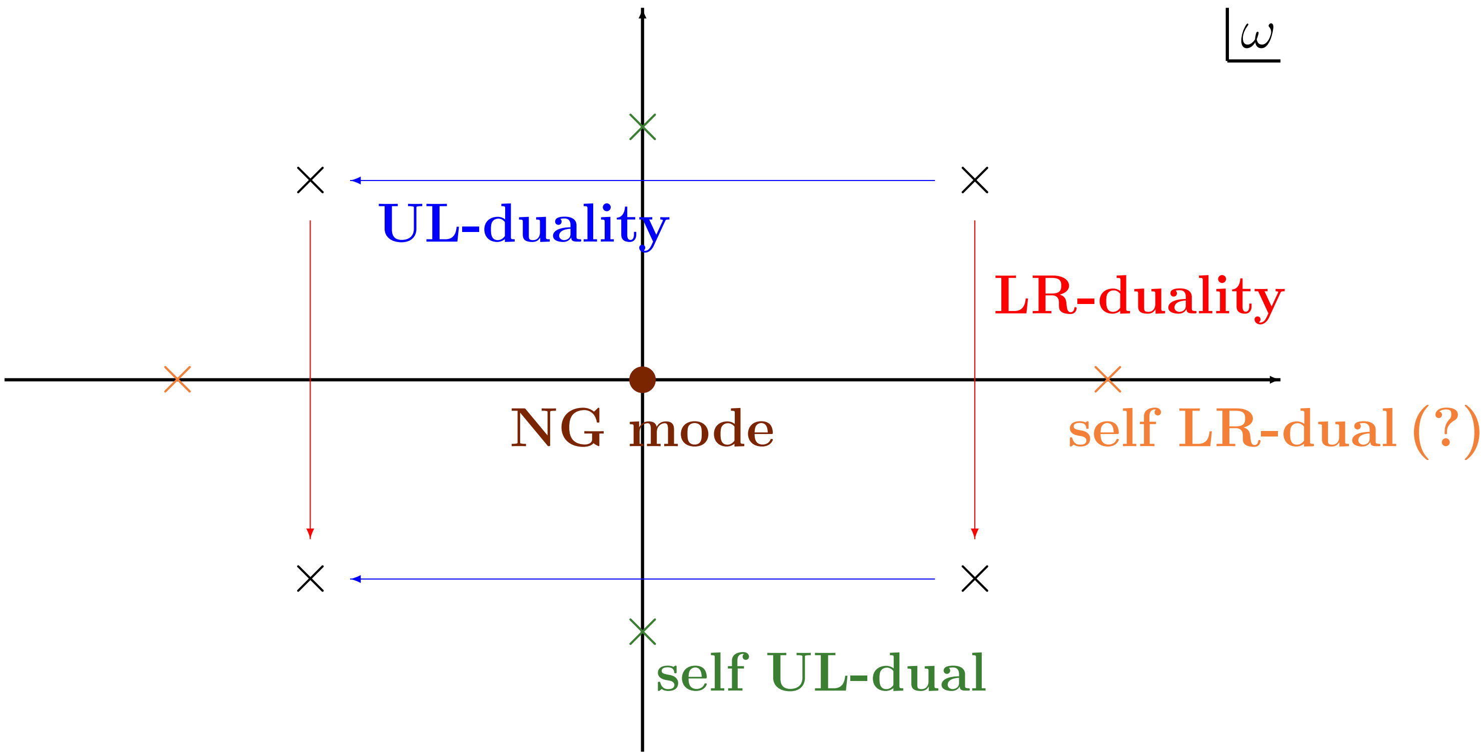
(iv) **a quartet of complex eigenvalues** $\pm\omega_\nu, \pm\omega_\nu^*$
with $\mathrm{Re}(\omega_\nu) \neq 0$**,** $\mathrm{Im}(\omega_\nu) \neq 0$**.**

(v) **a null eigenvalue.**

(no other possibilities)

(i)	$\cdots$	**physical solution**
(v)	$\cdots$	**NG-mode solution**
(ii–iv)	$\cdots$	**unphysical**

RPA solutions & dualities



★ **Dualities in full RPA space**

"RPA solutions" — not always cover the whole vector space

$\because$ $\mathsf{N}\,\mathsf{S}$ **may form Jordan blocks!** **cf. NG mode**

$\Rightarrow$ **dualities including Jordan bases** **($\rightarrow$ completeness)**

$$\boldsymbol{\xi}_1^{(\nu)} := \boldsymbol{x}_\nu\,, \quad \mathsf{S}\,\boldsymbol{\xi}_{k+1}^{(\nu)} = \omega_\nu\,\mathsf{N}\,\boldsymbol{\xi}_{k+1}^{(\nu)} + ic_k^{(\nu)}\,\mathsf{N}\,\boldsymbol{\xi}_k^{(\nu)} \quad (k = 1, \cdots, d_\nu - 1)$$

$$\rightarrow \textbf{Jordan block} \begin{pmatrix} \omega_\nu & ic_1^{(\nu)} & 0 & \cdots & 0 \\ 0 & \omega_\nu & ic_2^{(\nu)} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & ic_{d_\nu-1}^{(\nu)} \\ 0 & 0 & 0 & \cdots & \omega_\nu \end{pmatrix}$$

- **UL-duality**

$$\mathsf{S}\,\Sigma_x\boldsymbol{\xi}_{k+1}^{(\nu)*} = -\omega_\nu^*\,\mathsf{N}\,\Sigma_x\boldsymbol{\xi}_{k+1}^{(\nu)*} + ic_k^{(\nu)*}\,\mathsf{N}\,\Sigma_x\boldsymbol{\xi}_k^{(\nu)*}$$

$$\textbf{Jordan basis} \quad (\omega_\nu, \boldsymbol{\xi}_k^{(\nu)}; ic_k^{(\nu)}) \quad \rightarrow \quad (-\omega_\nu^*, \Sigma_x\boldsymbol{\xi}_k^{(\nu)*}; ic_k^{(\nu)*})$$

- **LR-duality**

Jordan block $\rightarrow (\mathsf{N}\,\bar{\boldsymbol{\xi}}_k^{(\nu)})^\dagger$ **: left Jordan basis of** $\mathsf{N}\,\mathsf{S}$ **for** ω_ν

$\leftrightarrow \mathsf{N}\,\bar{\boldsymbol{\xi}}_k^{(\nu)}$ **: right Jordan basis of** $\mathsf{S}\,\mathsf{N}$ **for** ω_ν^*

$\leftrightarrow \bar{\boldsymbol{\xi}}_k^{(\nu)}$ **: right Jordan basis of** $\mathsf{N}\,\mathsf{S}$ **for** ω_ν^*

choose so that $\left|\bar{\boldsymbol{\xi}}_k^{(\nu)\dagger}\,\mathsf{N}\,\boldsymbol{\xi}_{k'}^{(\nu')}\right| = \delta_{\nu\nu'}\,\delta_{kk'}$

$$\rightarrow \mathsf{S}\,\bar{\boldsymbol{\xi}}_{k-1}^{(\nu)} = \omega_\nu^*\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_{k-1}^{(\nu)} - ic_{k-1}^{(\nu)*}\,\frac{\boldsymbol{\xi}_{k-1}^{(\nu)\dagger}\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_{k-1}^{(\nu)}}{\boldsymbol{\xi}_k^{(\nu)\dagger}\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_k^{(\nu)}}\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_k^{(\nu)}$$

Jordan basis $\quad (\omega_\nu, \boldsymbol{\xi}_k^{(\nu)}; ic_k^{(\nu)}) \quad \rightarrow \quad \left(\omega_\nu^*, \bar{\boldsymbol{\xi}}_k^{(\nu)}; -ic_k^{(\nu)*}\,\frac{\boldsymbol{\xi}_k^{(\nu)\dagger}\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_k^{(\nu)}}{\boldsymbol{\xi}_{k+1}^{(\nu)\dagger}\,\mathsf{N}\,\bar{\boldsymbol{\xi}}_{k+1}^{(\nu)}}\right)$

with $k = d_\nu, d_\nu - 1, \cdots, 1$

cf. solution — $(\omega_\nu^*, \bar{\boldsymbol{\xi}}_{d_\nu}^{(\nu)})$

$\Rightarrow$ **2 types of dualities disclosed in entire RPA space!**

(— in deep relevance to most mathematical properties of RPA)

III. Decomposition of RPA space

$\nu \to$ index of Jordan block

$$\textbf{RPA space } \mathcal{V} = \bigoplus_{\nu} \mathcal{W}_\nu \,; \qquad \mathcal{W}_\nu := \left\{ \sum_{k=1}^{d_\nu} a_k \, \boldsymbol{\xi}_k^{(\nu)} \,;\, a_k \in \mathbf{C} \right\}$$

$$\textbf{subspace } \mathcal{W}_\nu \quad \leftrightarrow \quad \textbf{projector } \Lambda_\nu := \sum_{k=1}^{d_\nu} \frac{\boldsymbol{\xi}_k^{(\nu)} \, \bar{\boldsymbol{\xi}}_k^{(\nu)\dagger} \, \mathsf{N}}{\bar{\boldsymbol{\xi}}_k^{(\nu)\dagger} \, \mathsf{N} \, \boldsymbol{\xi}_k^{(\nu)}}$$

$$\Lambda_\nu \, \mathcal{W}_{\nu'} = \delta_{\nu\nu'} \, \mathcal{W}_\nu$$

$$\to \textstyle\sum_\nu \Lambda_\nu = 1 \ \textbf{(completeness)}$$

$\hookrightarrow$ respecting dualities

$$\mathcal{W}_{[\nu]} := \mathcal{W}_\nu \oplus \Sigma_x \mathcal{W}_\nu^* \oplus \bar{\mathcal{W}}_\nu \oplus \Sigma_x \bar{\mathcal{W}}_\nu^* \qquad \textbf{(cf. self dualities)}$$

$$\leftrightarrow \quad \Lambda_{[\nu]} \propto \Lambda_\nu + \Sigma_x \Lambda_\nu^* \Sigma_x + \mathsf{N} \Lambda_\nu^\dagger \mathsf{N} + \Sigma_x \mathsf{N} \Lambda_\nu^T \mathsf{N} \Sigma_x$$

$$\left(\Lambda_{[\nu]} = \Sigma_x \, \Lambda_{[\nu]}^* \, \Sigma_x = \mathsf{N} \, \Lambda_{[\nu]}^\dagger \, \mathsf{N} \,, \ \ \Lambda_{[\nu]} \, \Lambda_{[\nu']} = \delta_{[\nu],[\nu']} \, \Lambda_{[\nu]} \right)$$

- $d_{[\nu]} := \dim \mathcal{W}_{[\nu]} = \text{even}$ **(including self dual cases)**
- $\Lambda_{[\nu]}^\dagger \, \mathsf{S} \, \Lambda_{[\nu']} = \delta_{[\nu],[\nu']} \, \mathsf{S}_{[\nu]} \quad \to \quad \mathsf{S} = \sum_{[\nu]} \mathsf{S}_{[\nu]}$

$$\left(\mathsf{S}_{[\nu]} := \Lambda_{[\nu]}^\dagger \, \mathsf{S} \, \Lambda_{[\nu]} \ \cdots \ \textbf{inherits dualities} \right)$$

IV. Self dualities, Nambu-Goldstone mode

★ **Self dualities**

- **Self UL-duality :** $\boldsymbol{\xi}_k^{(\nu)} = -\Sigma_x \boldsymbol{\xi}_k^{(\nu)*} = \begin{pmatrix} \Xi^{(\nu,k)} \\ -\Xi^{(\nu,k)*} \end{pmatrix}$

  $\mathrm{Re}(\omega_\nu) = 0$ — **necessary & sufficient**

- **Self LR-duality :** $\boldsymbol{\xi}_k^{(\nu)} = \bar{\boldsymbol{\xi}}_k^{(\nu)}$

  $\mathrm{Im}(\omega_\nu) = 0$ & $d_\nu = 1$ — **necessary** ($\because$ $\bar{\boldsymbol{\xi}}_1^{(\nu)}$ **must be an eigenvector)**

  'normalizable' $\subset$ **'self LR-dual'**
  $(\boldsymbol{x}_\nu^\dagger \mathsf{N} \boldsymbol{x}_\nu > 0)$ $(\boldsymbol{x}_\nu^\dagger \mathsf{N} \boldsymbol{x}_\nu \neq 0)$

  $\hookrightarrow$ **self LR-dual Jordan block :** $\boldsymbol{\xi}_k^{(\nu)} = \bar{\boldsymbol{\xi}}_{d_\nu+1-k}^{(\nu)}$ $\cdots$ $d_\nu > 1$ **allowed**

- **Double self duality** $\cdots$ **intersection of both self dualities**

  $\omega_\nu = 0 \leftrightarrow$ **Nambu-Goldstone mode**

  ◦ **doubly self dual vector ?** — **impossible** ($\leftrightarrow$ $d_{[\nu]} = \mathrm{even} \neq 1$)

  ◦ **doubly self dual Jordan block ?** — **possible only if** $d_\nu = \mathrm{even}$

★ **NG mode & canonical variables from the duality viewpoint**

<u>**textbook formulae**</u>   <u>**in the present notation**</u>

(*e.g.* Ring-Schuck)

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} P \\ -P^* \end{pmatrix} = 0 \quad \to \quad \mathsf{S}\,\boldsymbol{x}_\nu = 0 \quad \textbf{with } \boldsymbol{x}_\nu = -\Sigma_x \boldsymbol{x}_\nu^*$$

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} Q \\ -Q^* \end{pmatrix} = -\frac{i\hbar}{M} \begin{pmatrix} P \\ P^* \end{pmatrix} \quad \to \quad \mathsf{S}\,\boldsymbol{\xi}_2^{(\nu)} = i c_1^{(\nu)}\,\mathsf{N}\,\boldsymbol{x}_\nu \quad \textbf{with } \boldsymbol{\xi}_2^{(\nu)} = -\Sigma_x \boldsymbol{\xi}_2^{(\nu)*}$$

$$\begin{pmatrix} P^* & P \end{pmatrix} \begin{pmatrix} Q \\ -Q^* \end{pmatrix} = \frac{\hbar}{i} \quad \to \quad \boldsymbol{x}_\nu^\dagger\,\mathsf{N}\,\boldsymbol{\xi}_2^{(\nu)} = -i \qquad (\to \ \boldsymbol{\xi}_2^{(\nu)} = \bar{\boldsymbol{x}}_\nu)$$

— **applicable only for $d_\nu = 2$ doubly self dual Jordan block !**

$\cdots$ **not always satisfied**

(*e.g.* a pair of $d_\nu = 1$ solutions, a $d_\nu = 4$ Jordan block)

V. Properties specific to positive-semidefinite stability matrix

MFA in self bound system → SSB unavoidable (c.m. motion)

→ **best-case scenario : positive-semidefinite** S (***i.e.*** $\omega_\nu \geq 0$ **for** $^\forall\nu$)

★ **If** S **is positive-semidefinite**

$$\Rightarrow \begin{cases} \textbf{physical solutions} \\ d_\nu \leq 2 \textbf{ NG modes,} \\ \quad \& \ d_\nu = 2 \textbf{ Jordan blocks must be doubly self dual} \end{cases}$$

∵) ∘ $\boldsymbol{x}_\nu \in \mathrm{Ker}(\mathsf{S}) \quad \leftrightarrow \quad \mathsf{S}\,\boldsymbol{x}_\nu = 0 \quad \leftrightarrow \quad \boldsymbol{x}_\nu$ **is a NG-mode solution**

∘ **if** $\boldsymbol{x}_\nu$ **is a solution of RPA eq. &** $\boldsymbol{x}_\nu \notin \mathrm{Ker}(\mathsf{S})$

$\rightarrow \quad \boldsymbol{x}_\nu^\dagger\,\mathsf{S}\,\boldsymbol{x}_\nu = \omega_\nu\,(\boldsymbol{x}_\nu^\dagger\,\mathsf{N}\,\boldsymbol{x}_\nu) > 0 \quad \rightarrow \quad$ **physical solution**

($\boldsymbol{x}_\nu^\dagger\,\mathsf{N}\,\boldsymbol{x}_\nu > 0$ **for** $\omega_\nu > 0$)

∘ **if** $\boldsymbol{x}_\nu$ **is a NG-mode solution associating with a Jordan block**

$\rightarrow \quad {}^\exists\boldsymbol{y}_\nu\,\big(\notin \mathrm{Ker}(\mathsf{S})\big),\ {}^\exists c_1^{(\nu)}\,(\neq 0) \quad \mathsf{S}\,\boldsymbol{y}_\nu = ic_1^{(\nu)}\,\mathsf{N}\,\boldsymbol{x}_\nu$

$\rightarrow \quad \boldsymbol{y}_\nu^\dagger\,\mathsf{S}\,\boldsymbol{y}_\nu = ic_1^{(\nu)}\,(\boldsymbol{y}_\nu^\dagger\,\mathsf{N}\,\boldsymbol{x}_\nu) > 0 \quad \rightarrow \quad \boldsymbol{y}_\nu^\dagger\,\mathsf{N}\,\boldsymbol{x}_\nu \neq 0$

$\rightarrow \quad d_\nu = 2$ **& doubly self dual** **q.e.d.**

↪ **"canonical variables" can be constituted** **(first complete proof ?)**

★ **Energy-weighted sum of excitation strengths**

exact theory : $\sum_\nu (E_\nu - E_0)\left|\langle\Psi_\nu|\hat{T}|\Psi_0\rangle\right|^2 = \frac{1}{2}\langle\Psi_0|[\hat{T},[\hat{H},\hat{T}]]|\Psi_0\rangle$

$\hat{T}(=\hat{T}^\dagger) = \sum_{k\ell} t_{k\ell}\, a_k^\dagger a_\ell$ **: transition op.**

RPA with positive definite S **:**

$$\Sigma_1 := \sum_\nu \omega_\nu \left|\langle\Phi_\nu|\hat{T}|\Phi_0\rangle\right|^2 = \frac{1}{2}\langle\Phi_0|[\hat{T},[\hat{H},\hat{T}]]|\Phi_0\rangle$$

($|\Phi_0\rangle$ **: MF state ,** $\langle\Phi_\nu|\hat{T}|\Phi_0\rangle$ **: transition amp. in RPA)**

$\hookrightarrow$ **influence of NG modes ?**

- **Revisiting proof for positive-definite case**

$$\mathsf{S}\,\mathsf{X} = \mathsf{N}\,\mathsf{X}\,\mathsf{N}\,\Omega\,, \quad \mathsf{X}^\dagger\,\mathsf{N}\,\mathsf{X} = \mathsf{N}\,;$$

$$\mathsf{X} := \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_D & \Sigma_x\boldsymbol{x}_1^* & \Sigma_x\boldsymbol{x}_2^* & \cdots & \Sigma_x\boldsymbol{x}_D^* \end{pmatrix}\,,\quad \Omega := \begin{pmatrix} \mathrm{diag}(\omega_\nu) & 0 \\ 0 & \mathrm{diag}(\omega_\nu) \end{pmatrix}$$

$$\to \quad \mathsf{N}\,\mathsf{S}\,\mathsf{N} = \mathsf{X}\,\Omega\,\mathsf{X}^\dagger \quad (\because\ \mathsf{X}\,\mathsf{N}\,\mathsf{X}^\dagger = \mathsf{N})$$

$$\to \quad \sum_\nu \omega_\nu \left|\langle\Phi_\nu|\hat{T}|\Phi_0\rangle\right|^2 = \frac{1}{2}\boldsymbol{t}^\dagger\,\mathsf{X}\,\Omega\,\mathsf{X}^\dagger\,\boldsymbol{t} = \frac{1}{2}\boldsymbol{t}^\dagger\,\mathsf{N}\,\mathsf{S}\,\mathsf{N}\,\boldsymbol{t} = \frac{1}{2}\langle\Phi_0|[\hat{T},[\hat{H},\hat{T}]]|\Phi_0\rangle$$

$$\boldsymbol{t} := \begin{pmatrix} t_{mi} & t_{mi}^* \end{pmatrix}^T$$

- **A pair of $d_\nu = 1$ solutions → trivial (no contribution to EWS)**
- **A $d_\nu = 2$ doubly self dual Jordan block**

$$\mathsf{S}\,\boldsymbol{p}_1 = 0\,,\ \ \mathsf{S}\,\boldsymbol{q}_1 = -i\zeta_1\,\mathsf{N}\,\boldsymbol{p}_1\,,\ \boldsymbol{q}_1^\dagger\,\mathsf{N}\,\boldsymbol{p}_1 = i \qquad (\boldsymbol{x}_1 \to \boldsymbol{p}_1\,,\ \boldsymbol{\xi}_2^{(1)} \to \boldsymbol{q}_1\,,\ c_1^{(1)} \to -\zeta_1)$$

$$\Rightarrow \quad \Lambda_{[1]} = \Lambda_1 = i(\boldsymbol{q}_1\,\boldsymbol{p}_1^\dagger - \boldsymbol{p}_1\,\boldsymbol{q}_1^\dagger)\,\mathsf{N}$$

$$\mathsf{S}\,\mathsf{X}' = \mathsf{N}\,\mathsf{X}''\,\mathsf{N}\,\Omega'\,, \qquad \mathsf{X}'^\dagger\,\mathsf{N}\,\mathsf{X}'' = \mathsf{N}\,;$$

$$\mathsf{X}' := \begin{pmatrix}\boldsymbol{p}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_D & \boldsymbol{q}_1 & \Sigma_x\boldsymbol{x}_2^* & \cdots & \Sigma_x\boldsymbol{x}_D^*\end{pmatrix}\,,$$

$$\mathsf{X}'' := \begin{pmatrix}i\boldsymbol{q}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_D & i\boldsymbol{p}_1 & \Sigma_x\boldsymbol{x}_2^* & \cdots & \Sigma_x\boldsymbol{x}_D^*\end{pmatrix}\,,$$

$$\Omega' := \begin{pmatrix}0 & & & \\ & \mathrm{diag}(\omega_\nu) & & \\ & & \zeta_1 & \\ & & & \mathrm{diag}(\omega_\nu)\end{pmatrix} \qquad (\nu = 2, 3, \cdots, D)$$

$$\to \quad \mathsf{N}\,\mathsf{S}\,\mathsf{N} = \mathsf{X}''\,\Omega'\,\mathsf{X}''^\dagger \qquad (\because\ \mathsf{X}''\,\mathsf{N}\,\mathsf{X}'^\dagger = \mathsf{X}'\,\mathsf{N}\,\mathsf{X}''^\dagger = \mathsf{N})$$

$$\rightarrow \quad \Sigma_1^{(+)} := \sum_{\nu\geq 2} \omega_\nu \left| \langle \Phi_\nu | \hat{T} | \Phi_0 \rangle \right|^2 = \frac{1}{2} \boldsymbol{t}^\dagger (1 - \Lambda_1) \, \mathsf{X}'' \, \Omega' \, \mathsf{X}''^\dagger (1 - \Lambda_1^\dagger) \, \boldsymbol{t}$$

$$= \frac{1}{2} \boldsymbol{t}^\dagger \, \mathsf{N} (1 - \Lambda_1^\dagger) \, \mathsf{S} (1 - \Lambda_1) \, \mathsf{N} \, \boldsymbol{t} \, ,$$

$$\Sigma_1^{(0)} := \frac{1}{2} \boldsymbol{t}^\dagger \Lambda_1 \, \mathsf{X}'' \, \Omega' \, \mathsf{X}''^\dagger \Lambda_1^\dagger \, \boldsymbol{t} = \frac{1}{2} \boldsymbol{t}^\dagger \, \mathsf{N} \, \Lambda_1^\dagger \, \mathsf{S} \, \Lambda_1 \, \mathsf{N} \, \boldsymbol{t}$$

$$\Sigma_1^{(0)} + \Sigma_1^{(+)} = \frac{1}{2} \langle \Phi_0 | [\hat{T}, [\hat{H}, \hat{T}]] | \Phi_0 \rangle$$

expansion of $\mathsf{N}\,\boldsymbol{t}$ by a complete set $\{\boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{x}_\nu, \Sigma_x \boldsymbol{x}_\nu^*\}$

$$\mathsf{N}\,\boldsymbol{t} = \alpha_p \, \boldsymbol{p}_1 + \alpha_q \, \boldsymbol{q}_1 + \sum_{\nu\geq 2} \left(\alpha_\nu \boldsymbol{x}_\nu - \alpha_\nu^* \, \Sigma_x \boldsymbol{x}_\nu^* \right)$$

$$\rightarrow \quad \Sigma_1^{(0)} = \frac{1}{2} \zeta_1 \, \alpha_q^2 = \frac{1}{2} \zeta_1 \left| \boldsymbol{p}_1^\dagger \, \boldsymbol{t} \right|^2 = \frac{1}{2} \zeta_1 \left| \langle \Phi_0 | [\hat{P}_1, \hat{T}] | \Phi_0 \rangle \right|^2$$

case of plural NG modes $(\# := K) \quad \rightarrow \quad \Sigma_1^{(0)} = \frac{1}{2} \sum_{\nu=1}^{K} \zeta_\nu \left| \langle \Phi_0 | [\hat{P}_\nu, \hat{T}] | \Phi_0 \rangle \right|^2$

cf. I. Stetcu and C.W. Johnson, P.R.C 67, 044315

equivalent formula, but not in terms of exp. value

($\rightarrow$ not so easy to apply)

- **Application to TRK sum rule**

  translational SSB — $\hat{P}_\nu = \mathbf{P}\,,\quad \zeta_\nu^{-1} = AM$

  ∘ $\hat{T}^{(E1)} = \dfrac{ZN}{A}(\mathbf{R}_p - \mathbf{R}_n)$ $\cdots$ **with c.m. correction**

  $\rightarrow\quad [\mathbf{P}, \hat{T}^{(E1)}] = 0 \quad\rightarrow\quad \Sigma_1^{(0)} = 0$

  $\rightarrow\quad \Sigma_1^{(+)} = \dfrac{1}{2}\langle\Phi_0|[\hat{T}^{(E1)}, [\hat{K}, \hat{T}^{(E1)}]]|\Phi_0\rangle = \dfrac{ZN}{2AM}$

  (discarding non-locality in charge-exchange int.)

  ∘ $\hat{T}^{(E1)'} = Z\,\mathbf{R}_p$ $\cdots$ **without c.m. correction**

  $\rightarrow\quad [\mathbf{P}, \hat{T}^{(E1)'}] = -i\,Z \quad\rightarrow\quad \Sigma_1^{(0)} = Z^2/2AM$

  $\rightarrow\quad \Sigma_1^{(+)} = \dfrac{1}{2}\langle\Phi_0|[\hat{T}^{(E1)'}, [\hat{K}, \hat{T}^{(E1)'}]]|\Phi_0\rangle - \Sigma_1^{(0)} = \dfrac{ZN}{2AM}$

  $\Sigma_1^{(0)}$ — **indeed carrying NG-mode contribution to EWS**

- **Application to rotational excitation in axially deformed nuclei**

  rotational SSB (under axial sym.) — $\hat{P}_\nu = J_x, J_y$ **,** $\zeta_\nu^{-1} = \mathcal{I}$

  $\hat{T}^{(X\lambda)}$ (λ**: rank,** X**: additional label *e.g.*** E/M)

  $\rightarrow \quad [J_{x,y}, \hat{T}_\mu^{(X\lambda)}] \propto \sqrt{\lambda(\lambda+1)}\,\delta_{|\mu|,1}\,\hat{T}_{\mu=0}^{(X\lambda)}$

  $\rightarrow \quad \Sigma_1^{(0)} = \dfrac{\lambda(\lambda+1)}{2\mathcal{I}}\left|\langle\Phi_0|\hat{T}_{\mu=0}^{(X\lambda)}|\Phi_0\rangle\right|^2$

  — **generalization of Hamamoto, N.P.A 177, 484**

  & Kurasawa, P.T.P. 64, 2055

  $$= \sum_{J'} \frac{J'(J'+1) - J(J+1)}{2\mathcal{I}} \left|\langle\Phi_0|\hat{T}_{\mu=0}^{(X\lambda)}|\Phi_0\rangle\right|^2 (J\,0\,\lambda\,0\,|\,J'\,0)^2$$

  rotational energy **transition strength**

  - ∘ **simple algebra, transparent result**
  - ∘ **not constrained to** $E2$ **cf. Hamamoto, Kurasawa**
  - ∘ **not constrained to** $J=0$ **initial state**

$$\begin{cases} \Sigma_1^{(0)} & \text{— intra-band EWS} \\ \Sigma_1^{(+)} & \text{— inter-band EWS} \end{cases}$$

($\Sigma_1^{(0)} + \Sigma_1^{(+)}$ — **full EWS**)

VI. Summary

Two types of "dualities" in the RPA space have been disclosed

⇒ complete analysis of RPA space **H.N., PTEP 2016, 063D02**

- **Complete classification of RPA solutions**
- **Decomposition of RPA space via duality-preserving projectors**
- **Self dualities — what they indicate, *etc.***

Properties specific to positive-semidefinite stability matrix:

- **(physical + NG-mode) solutions only,** $d_\nu \leq 2$

  $d_\nu = 2$ **Jordan block — doubly self dual** **H.N., PTEP 2016, 099101**

  → guaranteeing presence of canonical variable (first complete proof ?)

- **influence of NG modes on EWS of excitation strengths**

  → general formula in terms of exp. value of commutator at MF state

  applications ∘ TRK sum rule

  ∘ rotational excitations (cf. Hamamoto, Kurasawa)

  ⇒ reinforcing theoretical consistency of RPA framework !

H.N., arXiv: 1611.06623