Resonances in Lattice QCD

External Review on CCS, 2014/02/19

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Results of ρ meson decay by PACS-CS

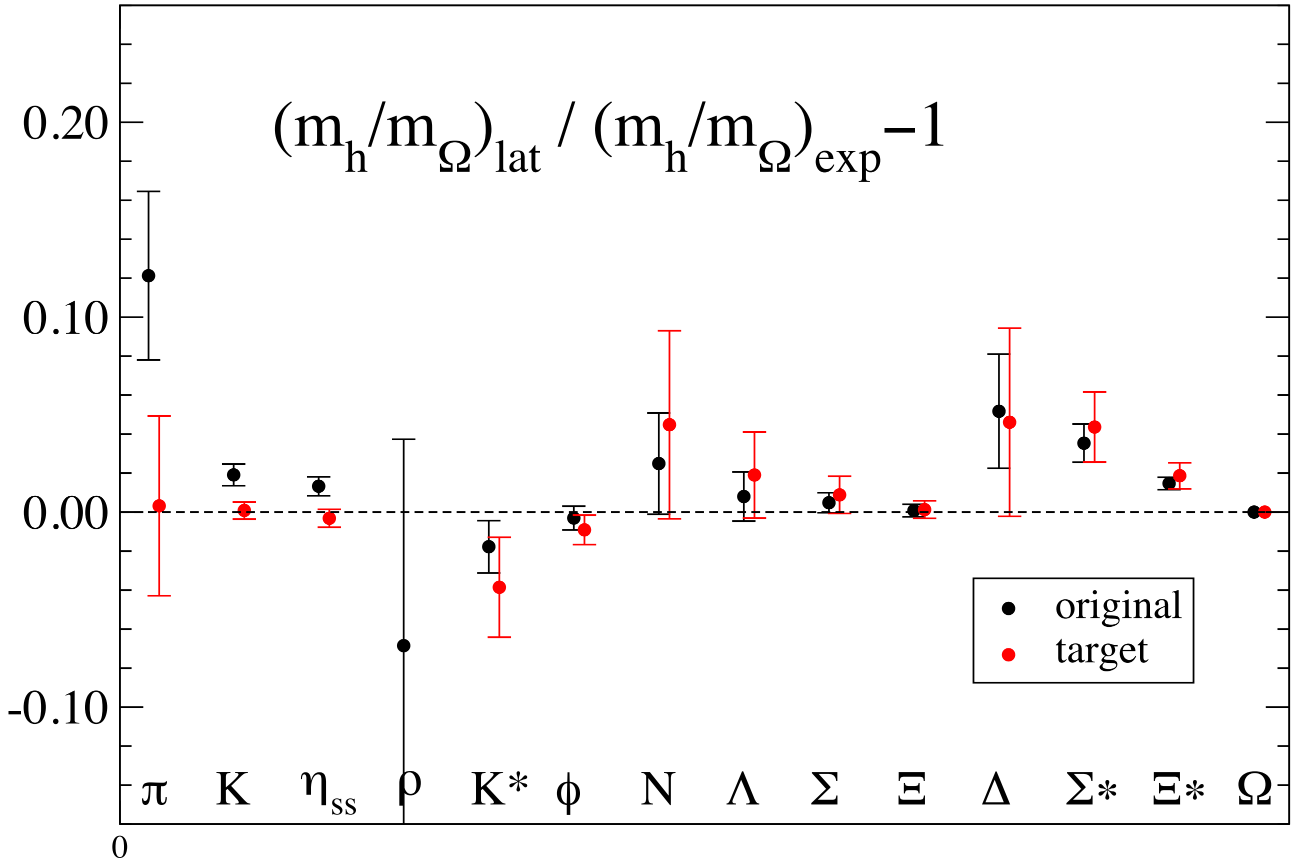
1. Introduction

Now, we can calculate hadron masses
at physical quark mass by lattice QCD.

But, it is only for stable particles ......

Hadron masses at physical quark mass

PACS-CS PRD81(2009)074503



: for unstable particle (*ρ* , *K** , *Δ*)
energies of ground states
on finite volume are plotted.

These are ***not resonance masses.***

Calculations of resonance mass
and decay width of unstable particles still remain.

Comment on time correlation function

Calculation of mass of stable particle

$\mathcal{O}(t)$: Heisenberg op. on lattice

$$\langle 0|\, \mathcal{O}(t)\, \mathcal{O}^\dagger(t_0)\, |0\rangle$$

$$= \sum_H \left|\langle 0|\mathcal{O}|H\rangle\right|^2 \cdot \mathrm{e}^{-E_H \cdot (t-t_0)}$$

$|H\rangle$: energy eigenstate

E_H : energy eigevalue

for $t >> t_0$

$$\sim \left|\langle 0|\mathcal{O}|H_0\rangle\right|^2 \cdot \mathrm{e}^{-E_0 \cdot (t-t_0)}$$

➡ energy of ground state : E_0 by exp-fit

for ρ meson (resonance state)

($\rho \to \pi\pi$ by strong interaction)

$$\langle 0|\ \rho^\dagger(t)\ \rho(0)\ |0\rangle = \sum_\alpha |\langle 0|\rho|\alpha\rangle|^2 \cdot \mathrm{e}^{-E_\alpha t}$$

: multi exp. form with $E_\alpha \in \mathbb{R}$

(: same form as for stable particle)

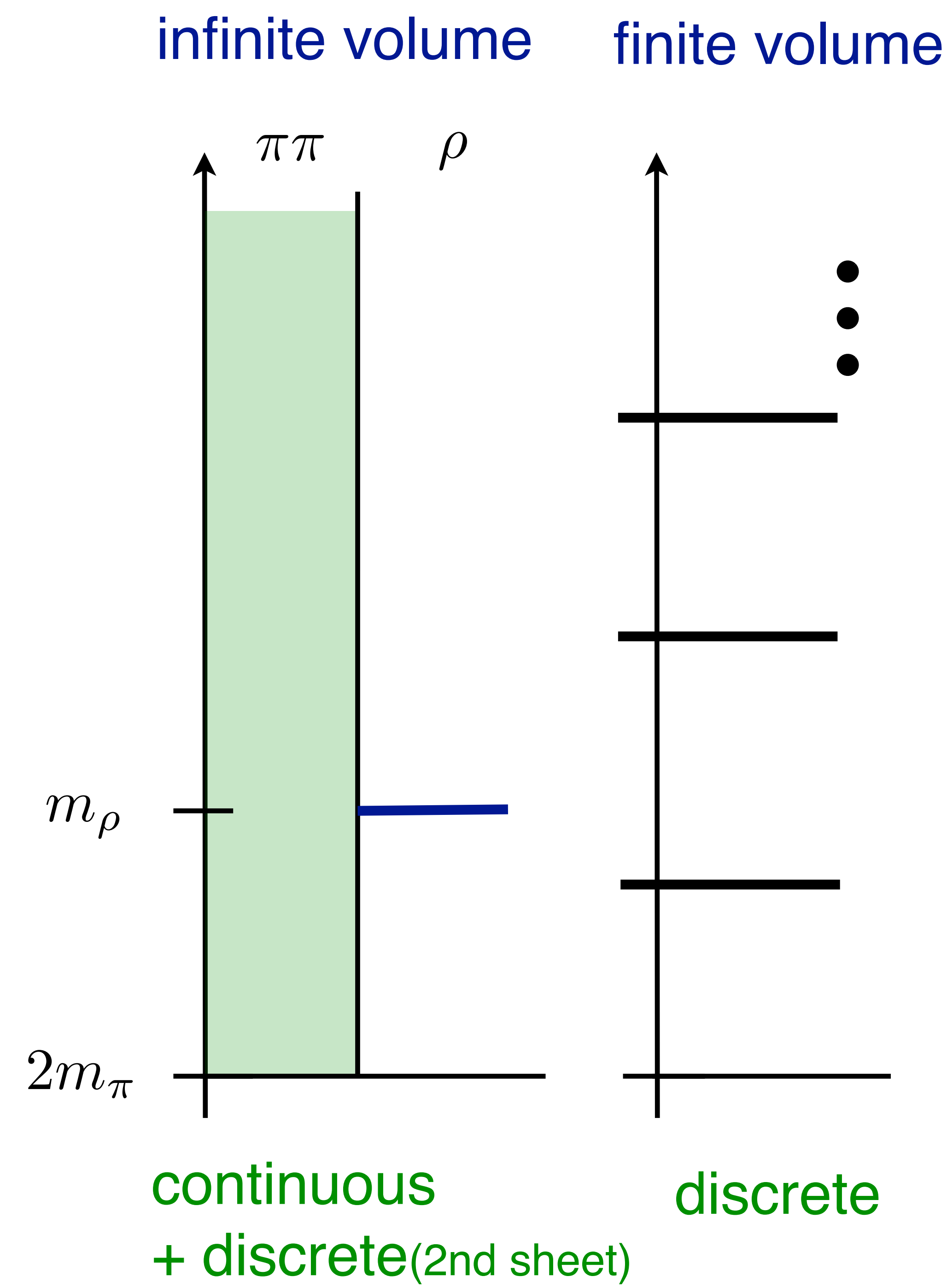
naive expectation :

for unstable particle

$$E = M + i\Gamma \qquad (M, \Gamma \in \mathbb{R})$$

time correlator $\sim \mathrm{e}^{-Mt} \cdot \mathrm{e}^{-i\Gamma t}$

: ***This is only true in infinite volume limit !***



Decay width can not be directly obtained from correlation function on the lattice.

Approach from momentum space

ex) ρ meson decay

Lattice calc :

$$\langle 0|\,\pi(p_1)\,\pi(p_2)\,\pi(p_3)\,\pi(p_4)\,|0\rangle \;\sim\; \sin\delta(k)\mathrm{e}^{i\delta(k)}$$

$$\delta(k) \;\rightarrow\; \Gamma$$ --- ???

Wrong !!

Lattice QCD is formulated on the Euclidian space-time.

$$p_j \quad : \quad \text{Euclid mom. on the lattice}$$

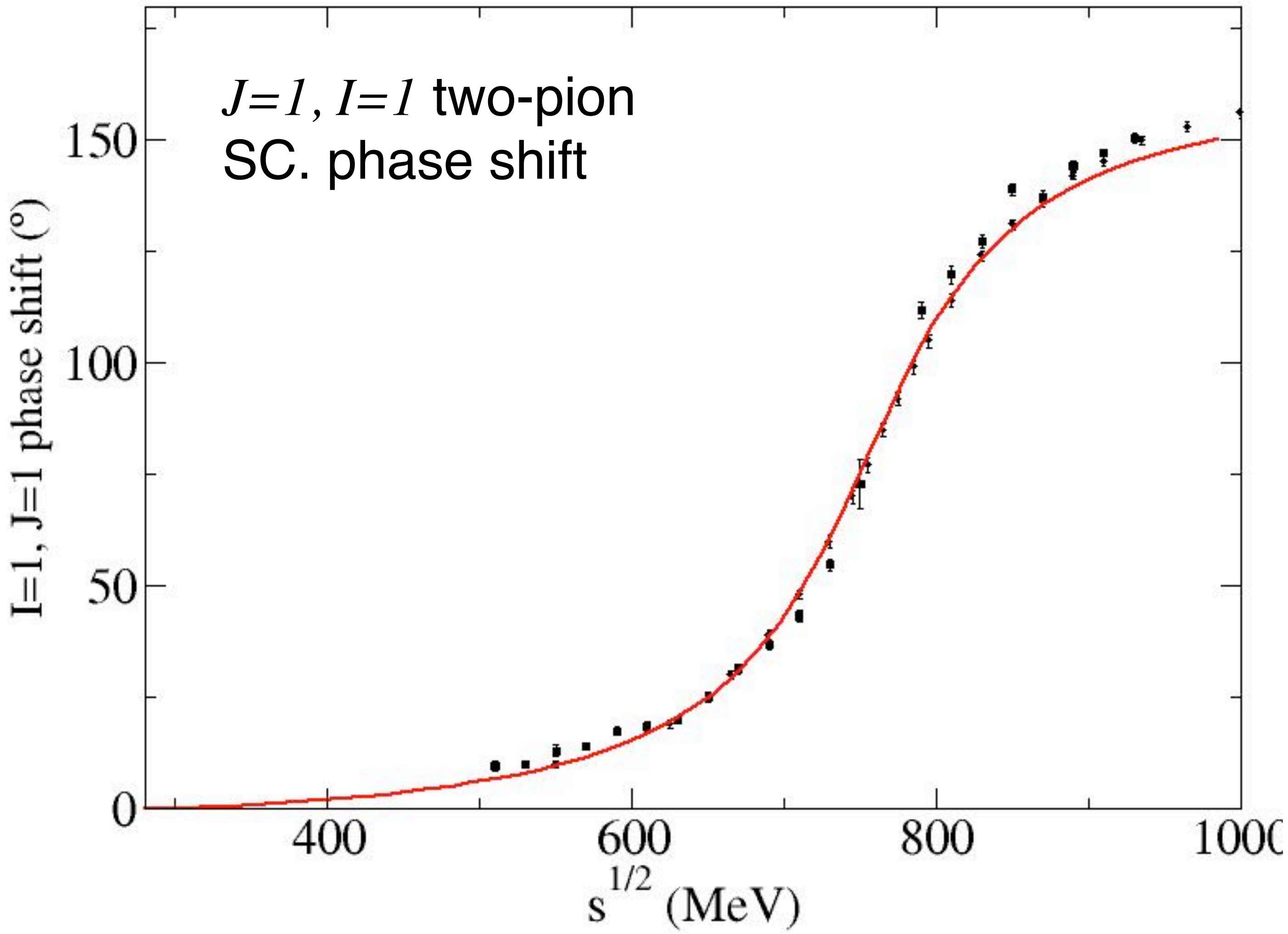
$\delta(k)$: un-physical

Is study of resonance from lattice QCD possible ?

YES !

We can calculate <u>scattering phase shift</u>
by ***finite size method****,*
and obtain information of resonance from it.

ex) ρ meson



→ existence of ρ

$$m_\rho = 775.7^{+4.3}_{-3.3}\ \mathrm{MeV}$$

$$\Gamma_\rho = 135.5^{+8.0}_{-9.0}\ \mathrm{MeV}$$

This is also possible
in the lattice QCD !!

Finite size method

Lüscher CMP105(86)153, NPB354(91)531.

Ex) for $\pi\pi$ system

In $L \times L \times L$ periodic box (: lattice)

Energy of π :

$$E = \sqrt{m_\pi^2 + k^2} \qquad k^2 = (2\pi/L)^2 \cdot n \,, \quad n \in \mathbb{Z}$$

Energy of $\pi\pi$:

$$E = 2\sqrt{m_\pi^2 + k^2} \qquad k^2 = (2\pi/L)^2 \cdot n \,, \quad n \notin \mathbb{Z}$$

(: discrete)

$$\frac{1}{\tan\delta_0(k)} = \frac{4\pi}{k} \cdot \frac{1}{L^3} \sum_{\mathbf{n}\in\mathbb{Z}^3} \frac{1}{p_n^2 - k^2} \qquad (\ \mathbf{p}_n = \mathbf{n} \cdot (2\pi)/L\ )$$

: SC. phase shift in infinite volume

: Lüscher's formula

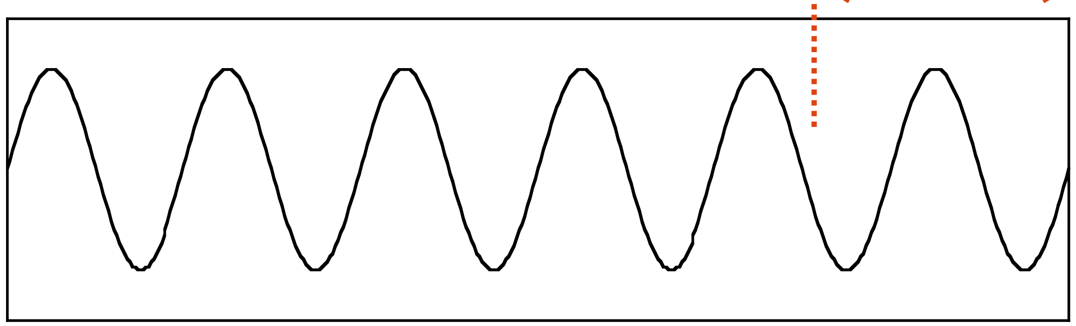
Energy of two-pion on the lattice $\Longrightarrow$ SC. phase shift in infinite volume $\Longrightarrow$ Resonance params.

Physical meaning of Lüscher's formula

for 1 dim case

No interacting two-pion
on the lattice

$\lambda = (2\pi)/k$



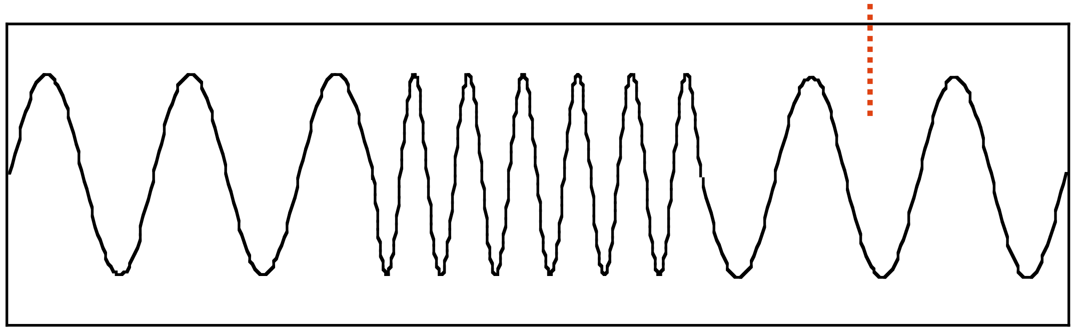
$-L/2$ $\qquad$ $L/2$

$$\boxed{kL = 2\pi \cdot n \qquad (\, n \in \mathbb{Z} \,)}$$

phase diff. = SC. phase shift : $\delta(k)$

Interacting two-pion
on the lattice

$\lambda = (2\pi)/k$



$-L/2$ $\qquad$ $L/2$

Interaction region

$$\boxed{kL + \underline{\delta(k)} = 2\pi \cdot n \quad (\, n \in \mathbb{Z} \,)}$$

$\delta(k)$: phase shift

Other possible methods :

1. From time correlation function

Extraction the resonance parameter
from a time correlation function by using an effective theory.

: strongly depends on an effective theory.

Recently : model indep. method at *L = huge*

U.-G. Meissner et.al, NPB846(2011)1

2. From spectrum density

V. Bernard et.al., JHEP 08(2008) 024

Calculations of energies on many lattice volumes
for very higher states are necessary.

: impossible in QCD !!

3. From "potential" extracted from BS-function

HAL collaboration 2012

Solving Shrodinger eq.
--> existence of resonance

Study of a resonance with these methods
has not been succeeded yet.

2. Previous works of ρ meson decay

$N_f = 2$: u, d $(m_u = m_d)$

$N_f = 2+1$: u, d, s $(m_u = m_d < m_s)$

$$\mathcal{L} = g_{\rho\pi\pi} \cdot \epsilon^{abc} \rho_\mu^a \pi^b \overleftrightarrow{\partial}_\mu \pi^c$$

exp. : $g_{\rho\pi\pi} = 5.878(22)$

$m_\rho = 775.5(0.4)\,\mathrm{MeV}$

$$\Gamma = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k_\rho^3}{m_\rho^2} = 146.4(1.1)\,\mathrm{MeV}$$

1) CP-PACS PRD76(07)094506.

$$N_f = 2\,(\mathrm{Wilson})\,,\ a = 0.21\,\mathrm{fm}\,,\ L = 2.5\,\mathrm{fm}\,,\ m_\pi = 330\,\mathrm{MeV}$$

$$g_{\rho\pi\pi} = 6.25(67)$$

2) M. Göckeler, R. Horsley, Y. Nakamura et.al. (QCDSF) arXiv:0810.5337 (Lat08)

$$N_f = 2\,(\mathrm{Wilson})\,,\ a = 0.072 - 0.084\,\mathrm{fm}\,,\ m_\pi = 240 - 810\,\mathrm{MeV}$$

$$g_{\rho\pi\pi} = 5.3^{+2.1}_{-1.5}$$

3) X. Feng, K. Jansen, D.B. Renner (ETMC) arXiv:0910.4871 (Lat09)

$$N_f = 2\,(\mathrm{tmQCD})\,,\ a = 0.086\,\mathrm{fm}\,,\ L = 2.1\,\mathrm{fm}\,,\ m_\pi = 391\,\mathrm{MeV}$$

$$g_{\rho\pi\pi} = 6.16(48)$$

4) ETMC

Lat10 (arXiv:1011.4288).
PRD83(2011)094505.

$N_f = 2\,(\mathrm{tmQCD}) \quad a = 0.079\,\mathrm{fm}$

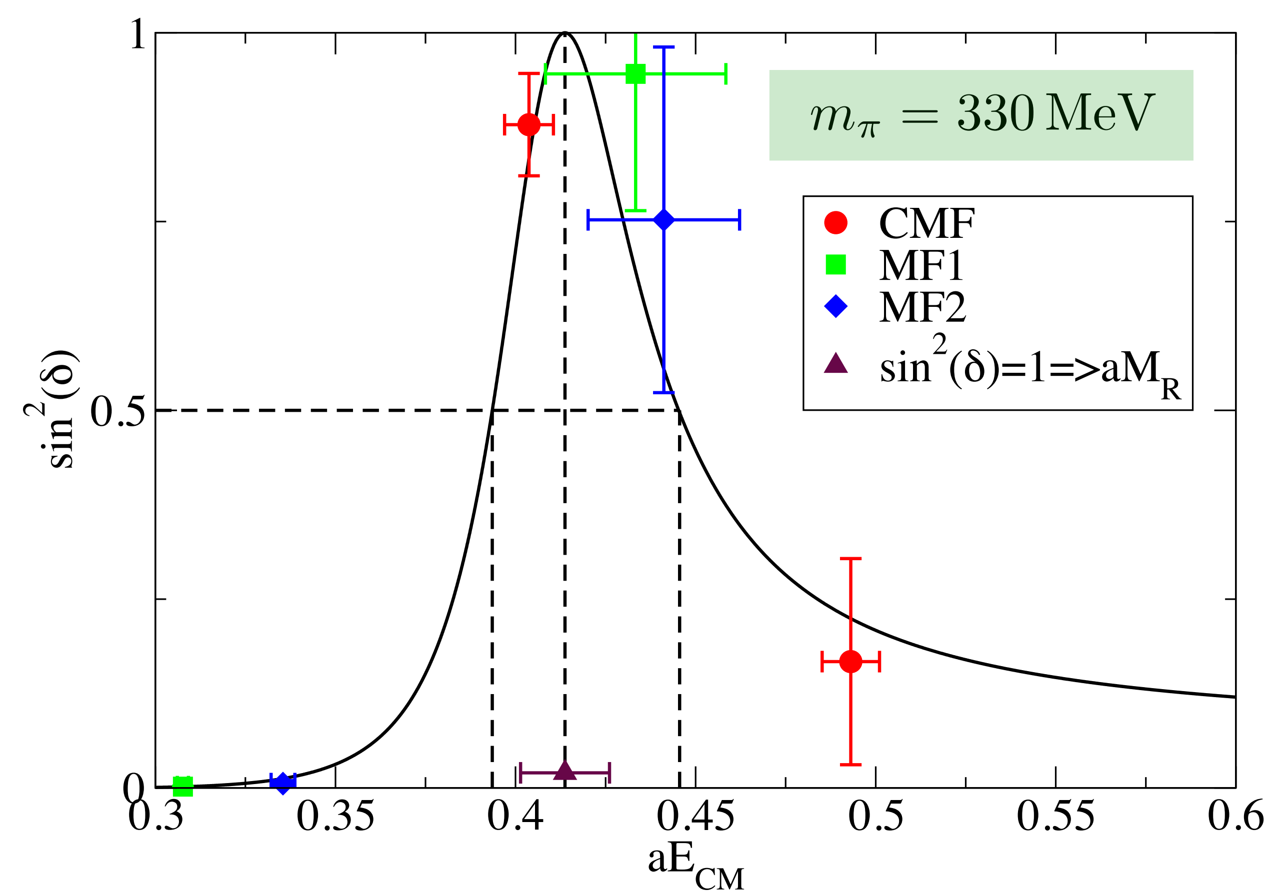
(Iso-spin sym. is broken : $\rho^+ \neq \rho^0$)

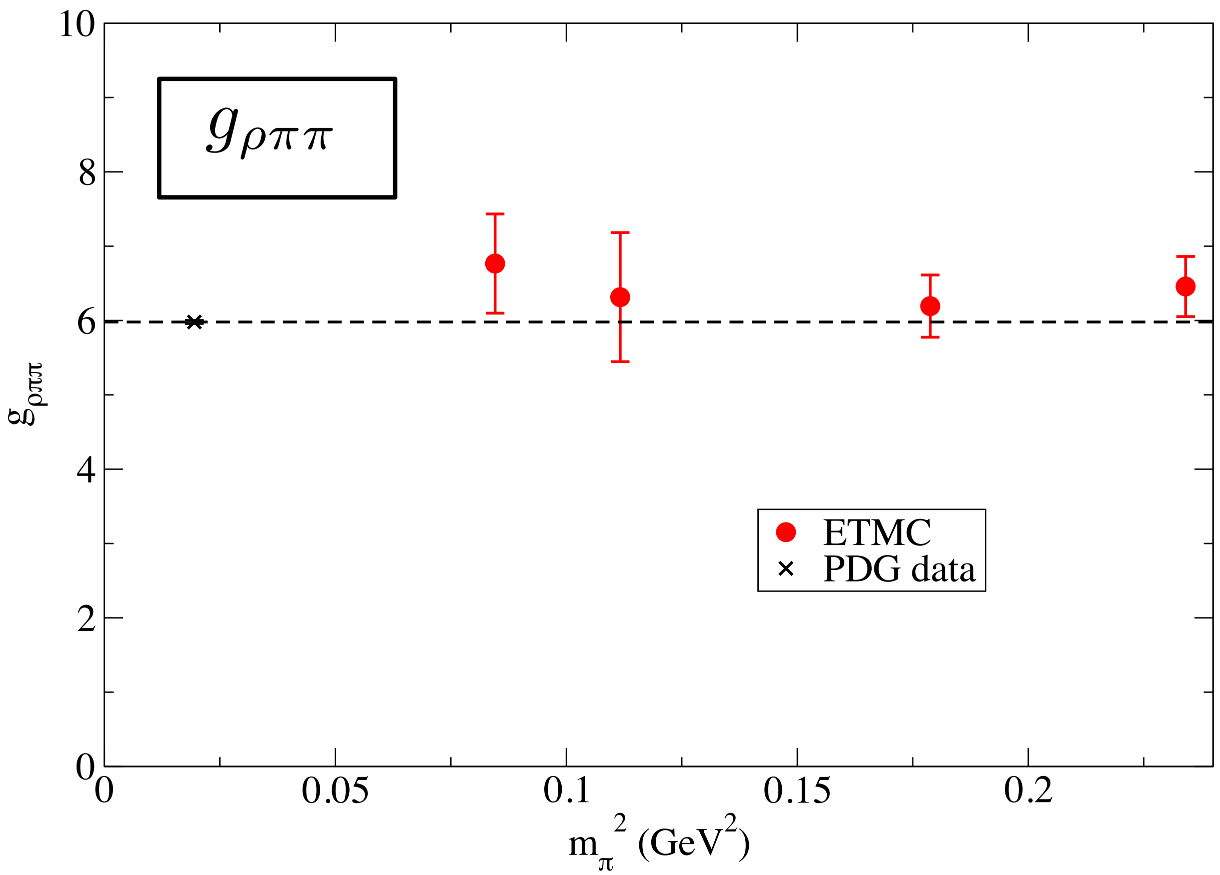
$L = 2.5\,\mathrm{fm}\,,\ m_\pi = 290, 330\,\mathrm{MeV}$

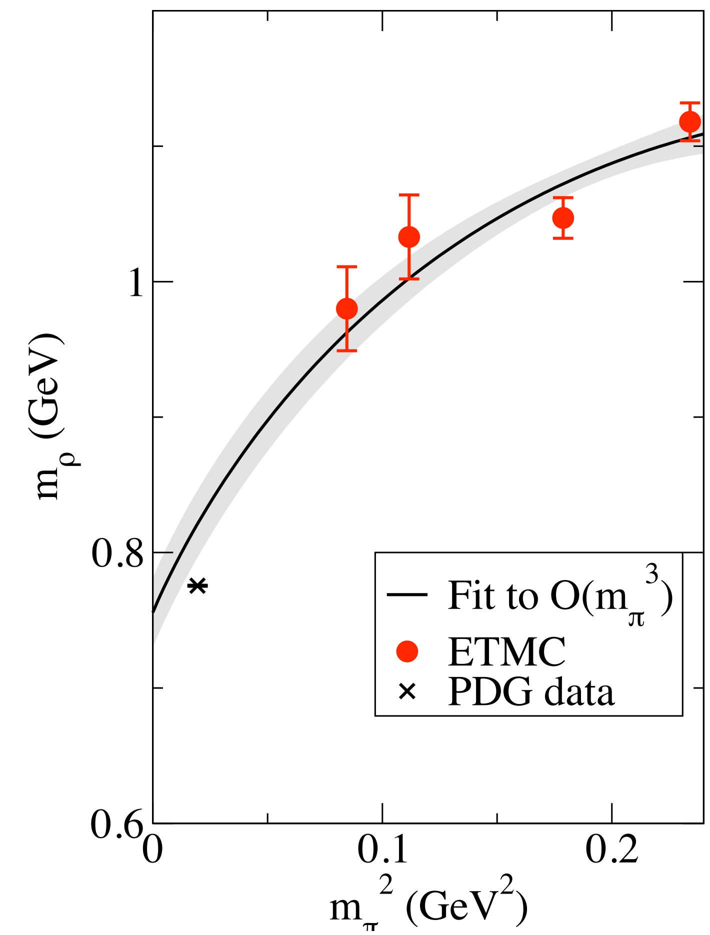
$L = 1.9\,\mathrm{fm}\,,\ m_\pi = 420, 480\,\mathrm{MeV}$

exp. : $g_{\rho\pi\pi} = 5.878(14)$

$m_\rho = 775.49(34)\,\mathrm{MeV}$







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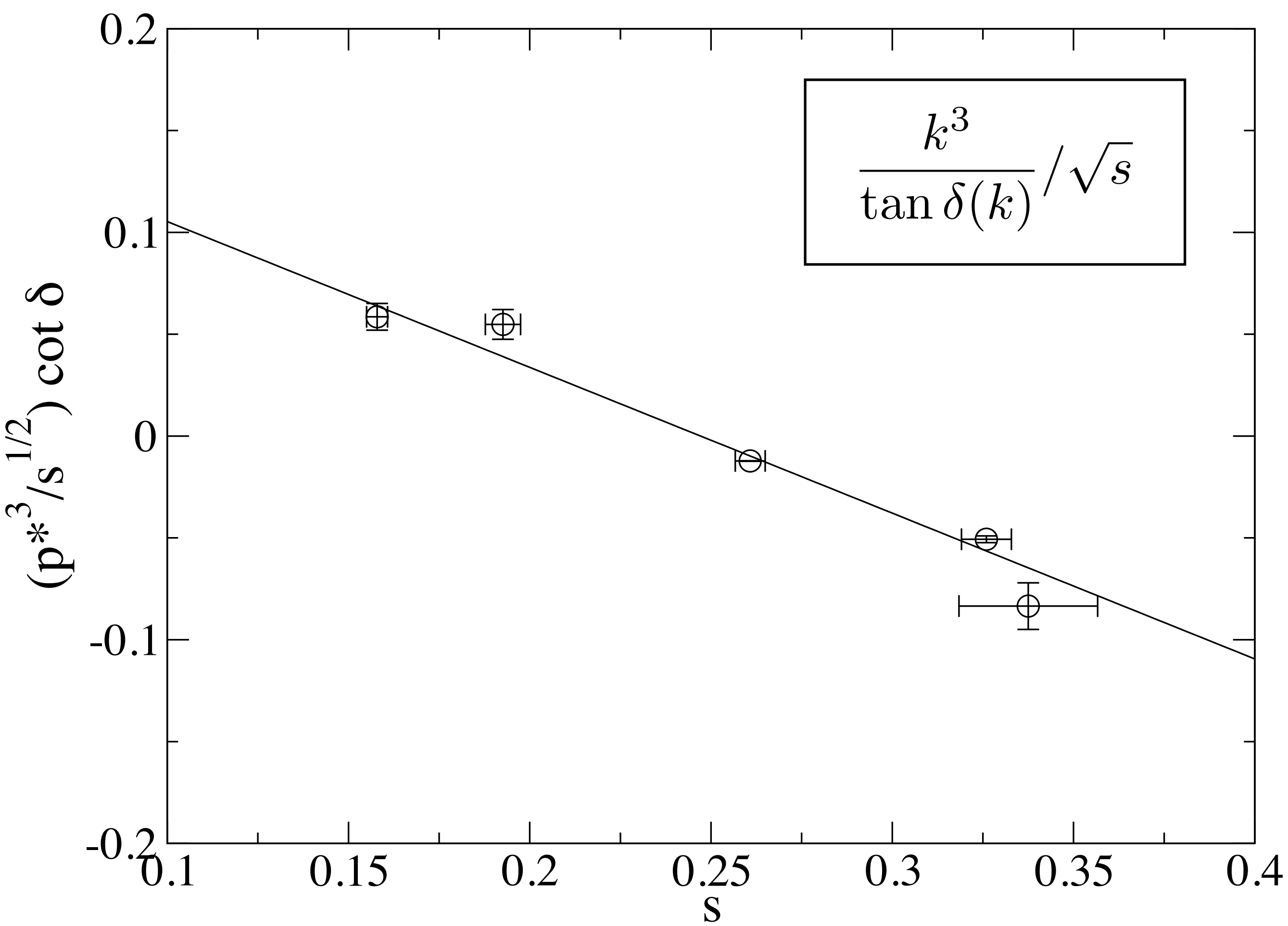
5) C.B. Lang et.al. PRD84(2011)054503.

exp. : $g_{\rho\pi\pi} = 5.878(14)$

$m_\rho = 775.49(34)\,\mathrm{MeV}$

$N_f = 2\,(\mathrm{Wilson}) \quad a = 0.124\,\mathrm{fm}$

$L = 2\,\mathrm{fm}\,,\ m_\pi = 266\,\mathrm{MeV}$



$$\frac{k^3}{\tan\delta(k)}/\sqrt{s} = \frac{6\pi}{g_{\rho\pi\pi}^2}\cdot(m_\rho^2 - s)$$

$g_{\rho\pi\pi} = 5.13(20)$

$m_\rho = 792(7)(8))\,\mathrm{MeV}$

at $m_\pi = 266\,\mathrm{MeV}$

2. Results of ρ meson decay (2009 - 2011)

PRD84(2011)094505.

Gauge conf. :

$N_f = 2 + 1\,(\mathrm{Wilson})$, Iwasaki Gauge action

$a = 0.091\,\mathrm{fm}$, $L = 2.9\,\mathrm{fm}$

$m_\pi = 300\,,\ 410\ \mathrm{MeV}$

generated by PACS-CS col.
PRD79(2009)034503.

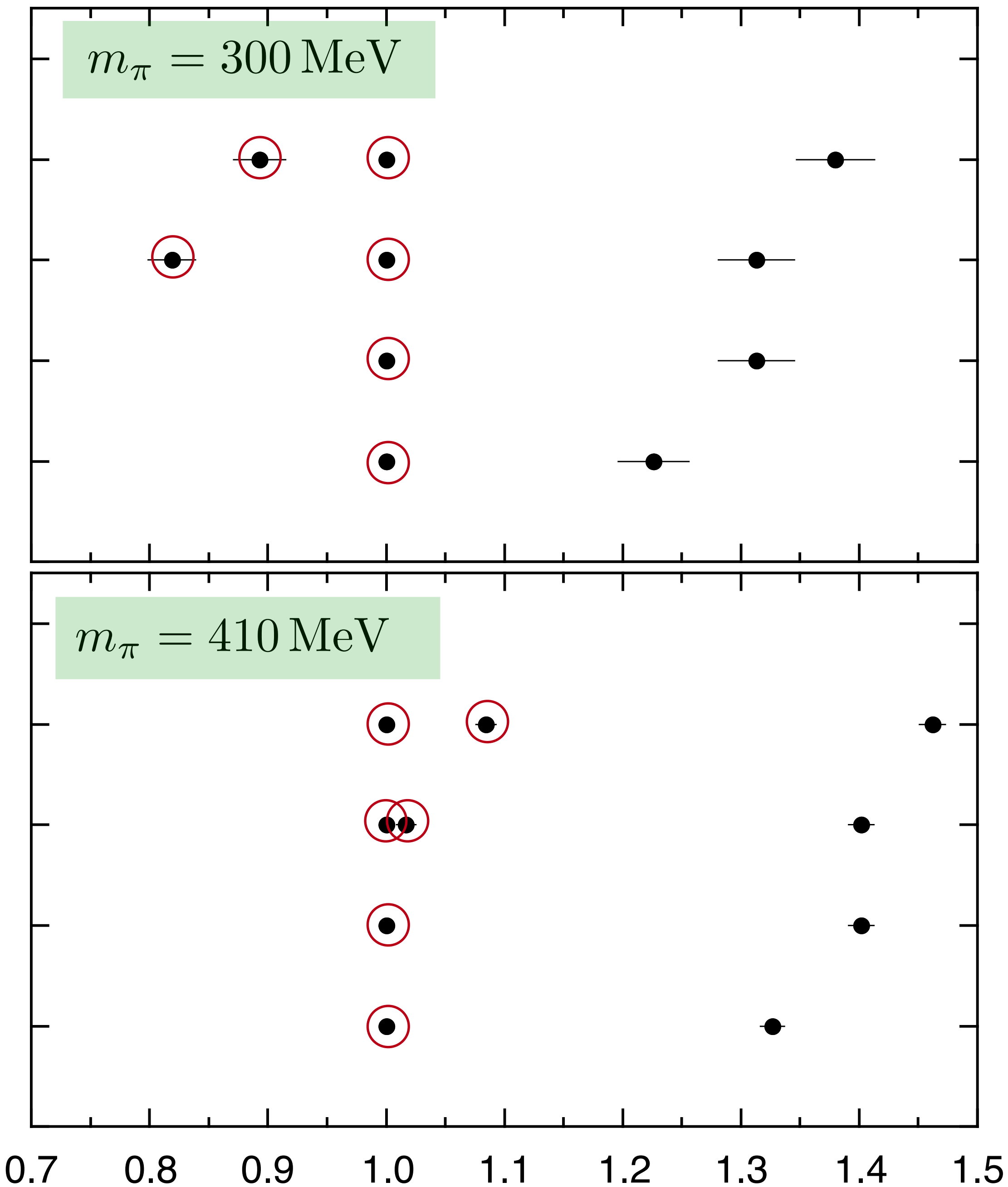
Computer : PACS-CS

Calc. points

We consider 4 irreps. : $(0,0,0)\,\mathbf{T}_1\,,\ (0,0,1)\,\mathbf{E}\,,\ (0,0,1)\,\mathbf{A}_2\,,\ (1,1,0)\,\mathbf{B}_1$

on 3 momentuma frames with $\mathbf{P}_{\mathrm{tot}} = (0,0,0)\,,(0,0,1)\,,(1,1,0)$

$\sqrt{s}/m_\rho$ *(ignoring particle int.)*



	ground	1st. Ex.
$(1,1,0)\,\mathbf{B}_1$	$\pi(1,1,0)\pi(0,0,0)$	$\rho_{1+2}(1,1,0)$
$(0,0,1)\,\mathbf{A}_2$	$\pi(0,0,1)\pi(0,0,0)$	$\rho_3(0,0,1)$
$(0,0,1)\,\mathbf{E}$	$\rho_{1,2}(0,0,1)$	$\pi(1,0,1)\pi(-1,0,0)$
$(0,0,0)\,\mathbf{T}_1$	$\rho_{1,2,3}(0,0,0)$	$\pi(0,0,1)\pi(0,0,-1)$

	ground	1st. Ex.
$(1,1,0)\,\mathbf{B}_1$	$\rho_{1+2}(1,1,0)$	$\pi(1,1,0)\pi(0,0,0)$
$(0,0,1)\,\mathbf{A}_2$	$\rho_3(0,0,1)$	$\pi(0,0,1)\pi(0,0,0)$
$(0,0,1)\,\mathbf{E}$	$\rho_{1,2}(0,0,1)$	$\pi(1,0,1)\pi(-1,0,0)$
$(0,0,0)\,\mathbf{T}_1$	$\rho_{1,2,3}(0,0,0)$	$\pi(0,0,1)\pi(0,0,-1)$

Finite size formula

M. Lüscher, NPB354(1991)531.
K.Rummainnen and S.Gottlib, NPB450(1995)397.
ETMC, arXiv:1011.5288.

Relation between $\delta(k)$ and E

$$\frac{1}{\tan\delta(k)} = \begin{cases} V_{00} & \text{for } (0,0,0)\ \mathbf{T}_1 \\ V_{00} - V_{20} & \text{for } (0,0,1)\ \mathbf{E} \\ V_{00} + 2\cdot V_{20} & \text{for } (0,0,1)\ \mathbf{A}_2 \\ V_{00} - V_{20} + \sqrt{6}\cdot V_{22} & \text{for } (1,1,0)\ \mathbf{B}_1 \end{cases}$$

$$V_{lm}(k;\mathbf{P}_{\mathrm{tot}}) = \frac{1}{\gamma q^{l+1}} \frac{1}{\pi^{3/2}\sqrt{2l+1}} \mathrm{e}^{-im\pi/4} \cdot Z_{lm}(1;q;\mathbf{d})$$

(: Real)

$$\sqrt{E^2 - P_{\mathrm{tot}}^2} = \sqrt{s} = 2\sqrt{m_\pi^2 + k^2}$$

$$q = kL/(2\pi) \qquad \mathbf{d} = \mathbf{P}_{\mathrm{tot}} L/(2\pi)$$

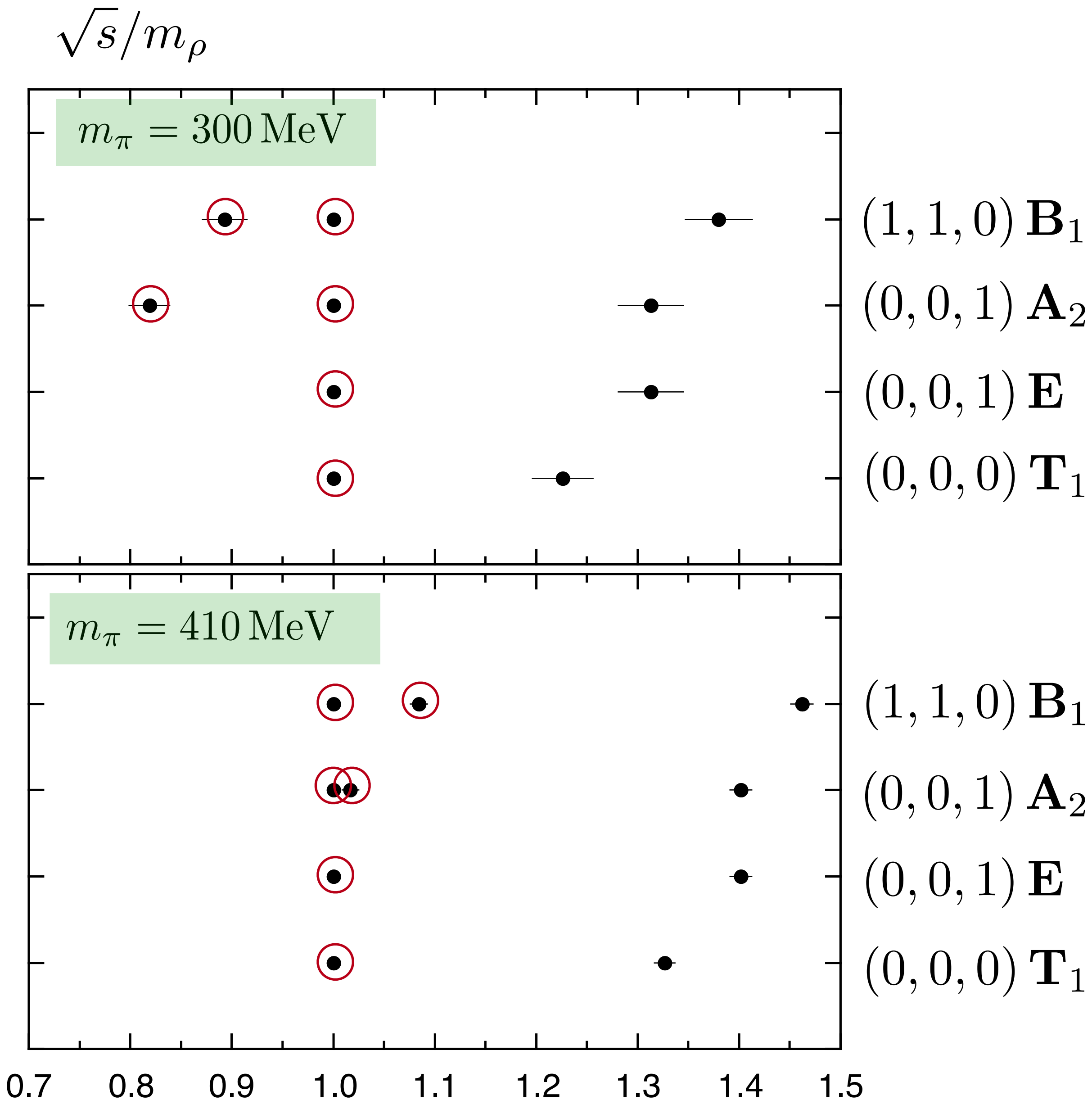
$$Z_{lm}(s;q;\mathbf{d}) = \sum_{\mathbf{r}\in D(\mathbf{d})} \mathcal{Y}_{lm}(\mathbf{r}) \cdot (|\mathbf{r}|^2 - q^2)^{-s}$$

$$D(\mathbf{d}) = \{\, \mathbf{r} \,|\, \mathbf{r} = \hat{\gamma}^{-1}(\mathbf{n} + \mathbf{d}/2)\ ,\ \mathbf{n}\in\mathbb{Z}^3 \,\}$$

$$\hat{\gamma}^{-1}\ :\ \text{Inv. Lorentz boost with } \gamma = E/\sqrt{s}$$

For $(0,0,1)\,\mathbf{A}_2$ and $(1,1,0)\,\mathbf{B}_1$

For $(0,0,0)\,\mathbf{T}_1$ and $(0,0,1)\,\mathbf{E}$
energies of ground state are extracted from correlation function of ρ meson as usual calculations of hadron masses.



Energies are extracted by variational method.

$$\mathcal{O}_1(t) = \rho(\mathbf{p}, t)$$

$$\mathcal{O}_2(t) = \pi^-(\mathbf{p}, t)\pi^+(\mathbf{0}, t) - \pi^+(\mathbf{p}, t)\pi^-(\mathbf{0}, t)$$

$$\text{for} \quad \mathbf{p} = (0,0,1)\,,\ (1,1,0)$$

$$G_{ij}(t) = \langle\, \mathcal{O}_i^\dagger(t)\ \mathcal{O}_j(0)\,\rangle \qquad (\ :\ 2\times 2 \quad \text{matrix}\ )$$

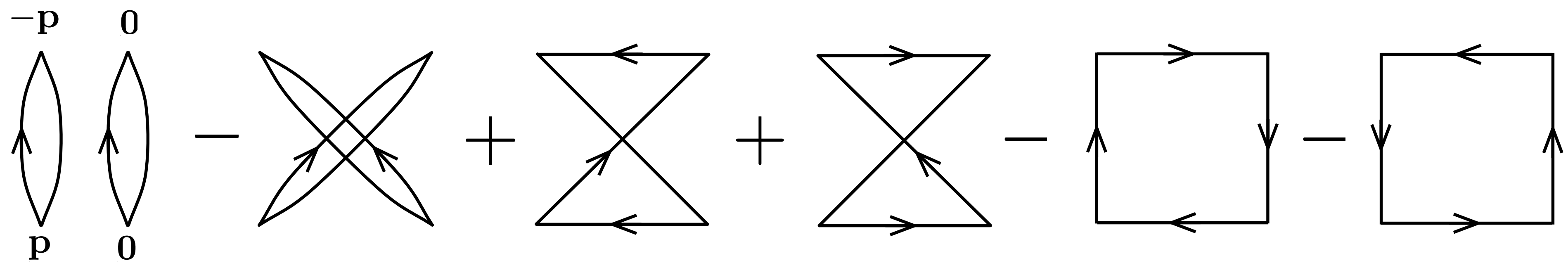
Assuming 2 states dominant for $t \gg t_s$

$$\mathrm{Ev}[\ G(t)\ G^{-1}(t_0)\ ]_\alpha = \mathrm{e}^{-W_\alpha\cdot(t-t_0)} \qquad (\ \alpha = 1,2\ )$$

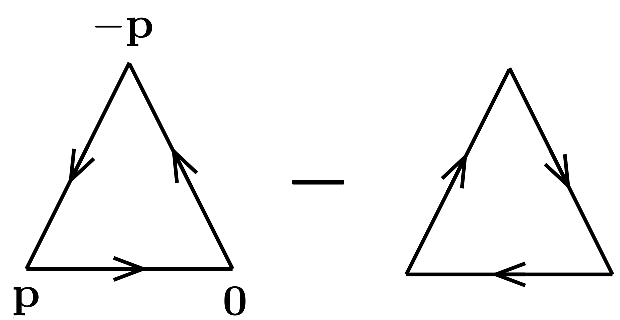
$$W_\alpha \ :\ \ \text{energy eigenvalue}$$

Calc. of $G(t)$

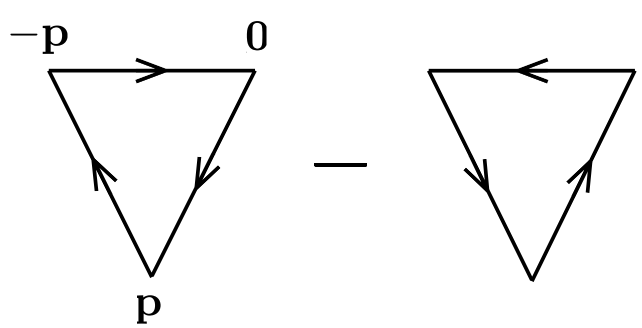
$$G_{\pi\pi\to\pi\pi}(t) =$$

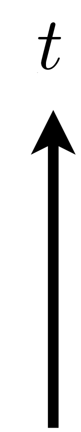


$$G_{\pi\pi\to\rho}(t) =$$



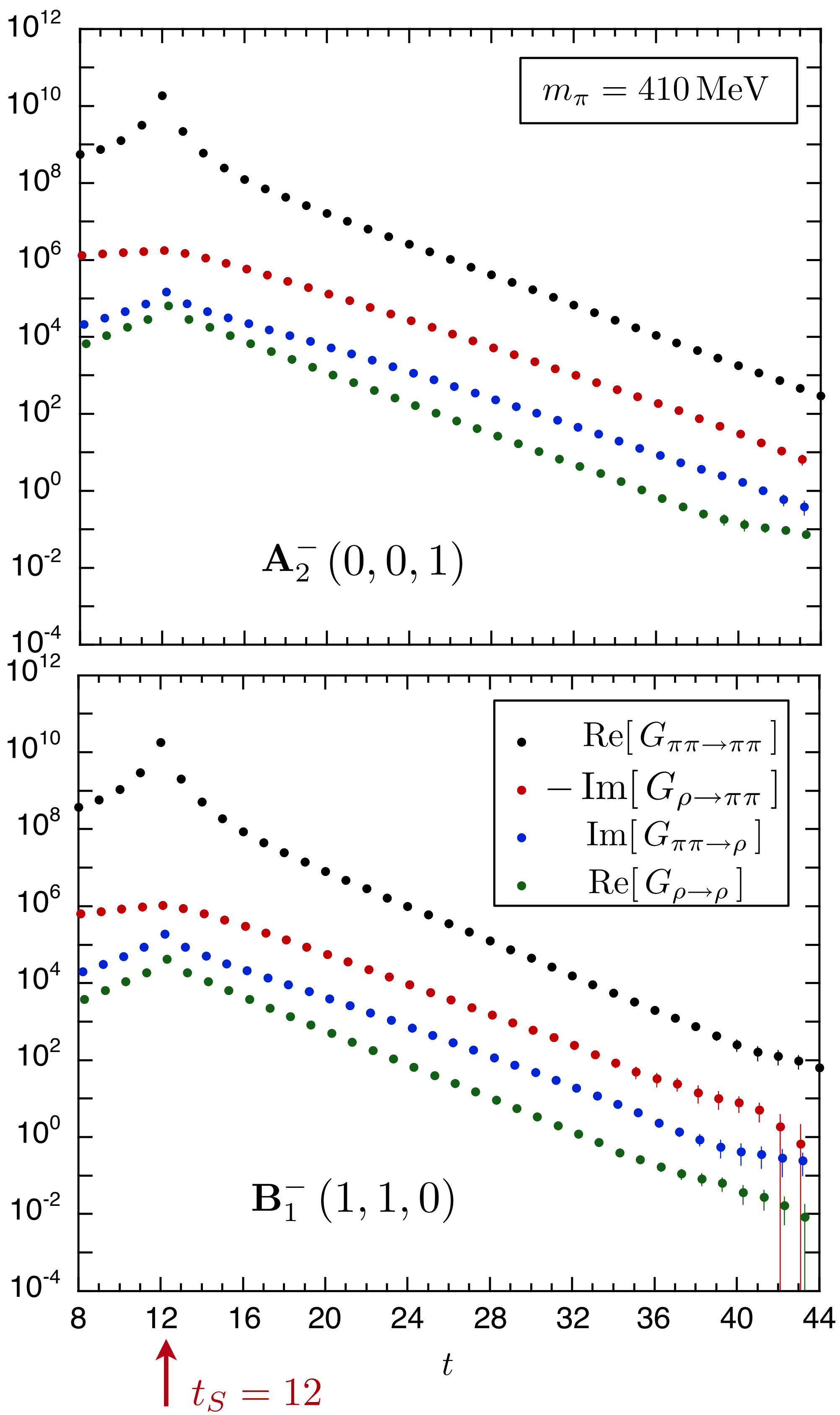
$$G_{\rho\to\pi\pi}(t) =$$



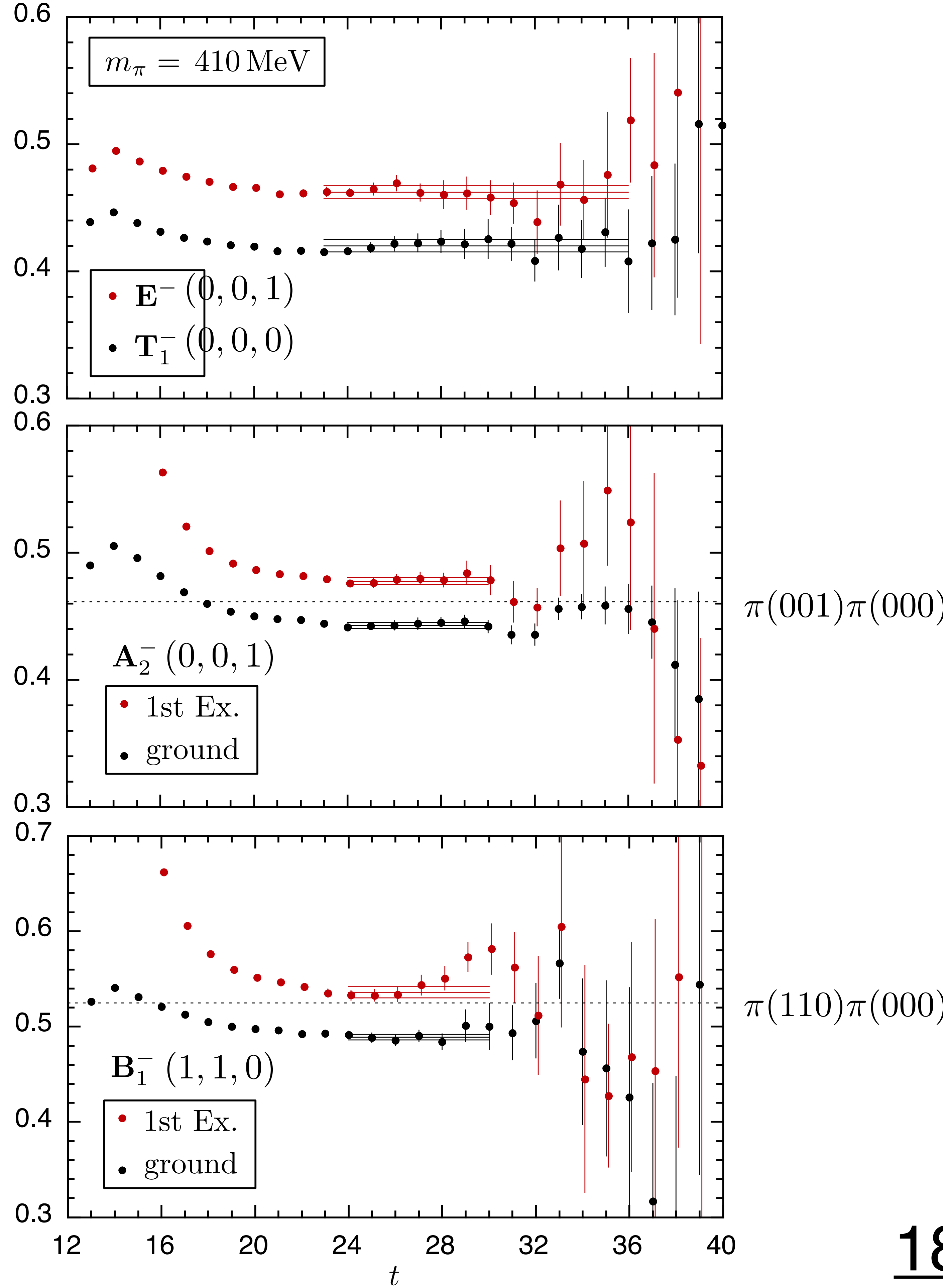


These are calculated
by the stochastic source and the source method.

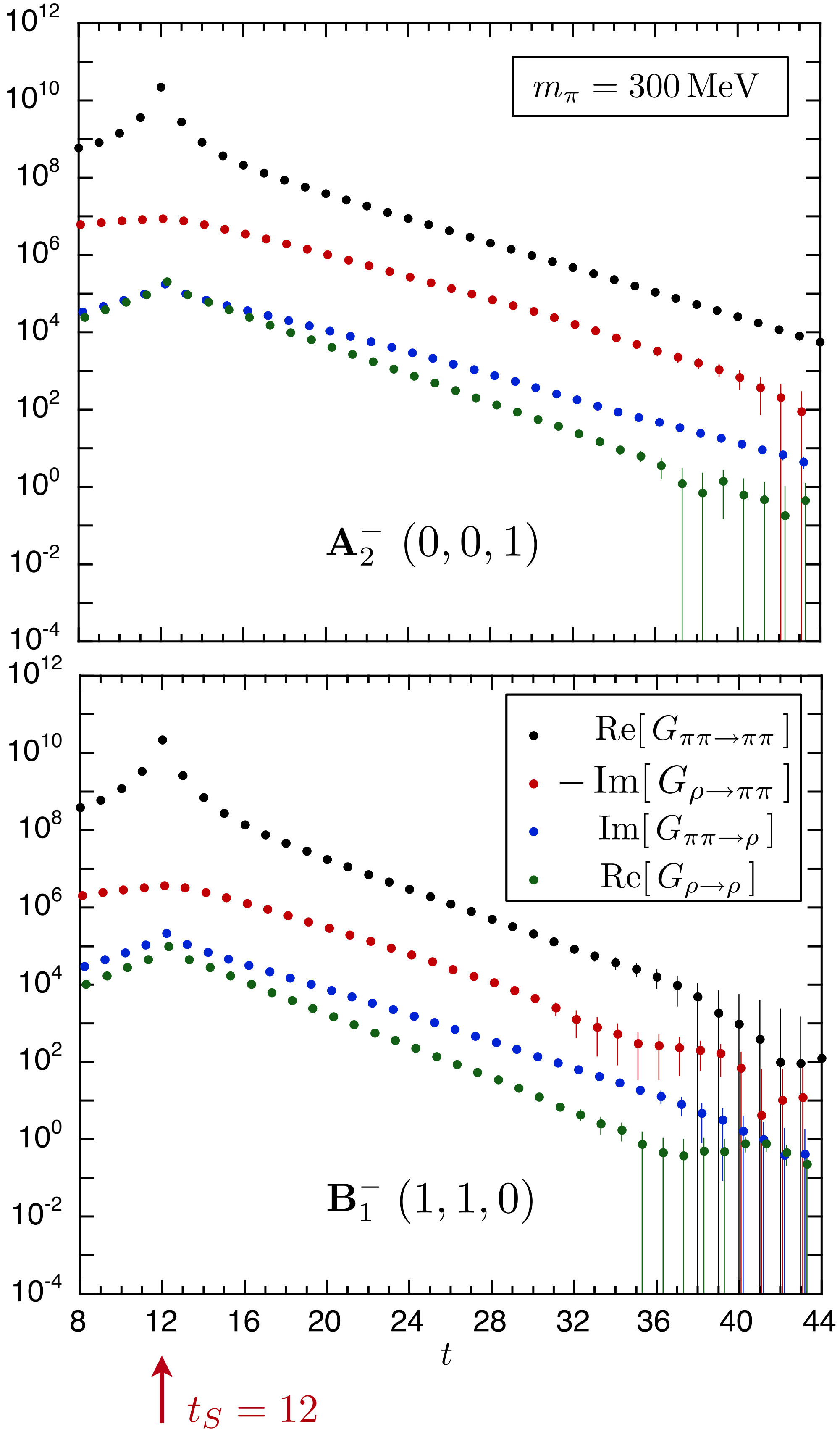
Results at $m_\pi = 410\,\mathrm{MeV}$



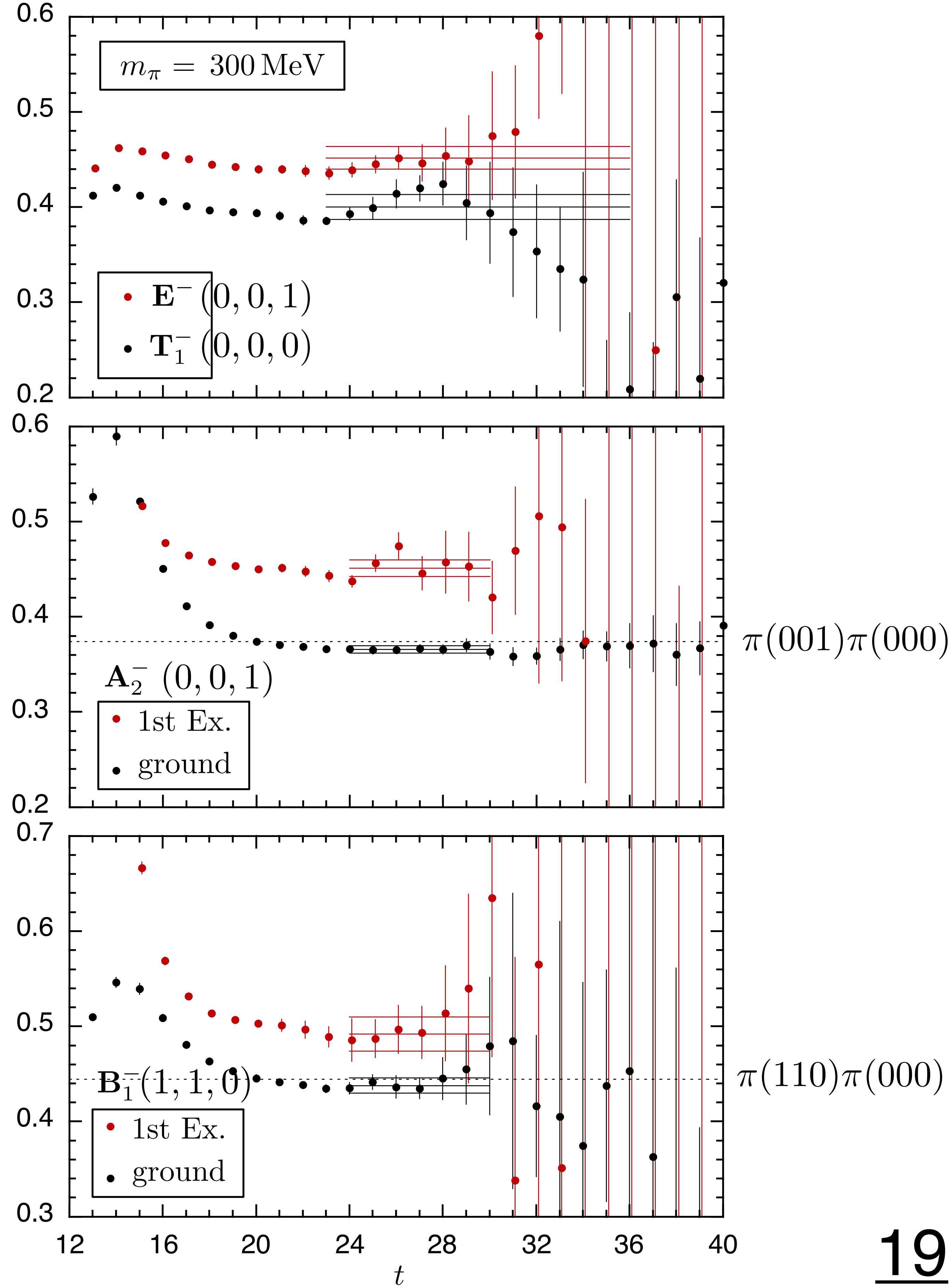
effective mass $\log G(t)/G(t+1) \sim E$



Results at $m_\pi = 300\,\mathrm{MeV}$



effective mass $\log G(t)/G(t+1) \sim E$



Results of SC. phase shift

exp. : $g_{\rho\pi\pi} = 5.878(14)$

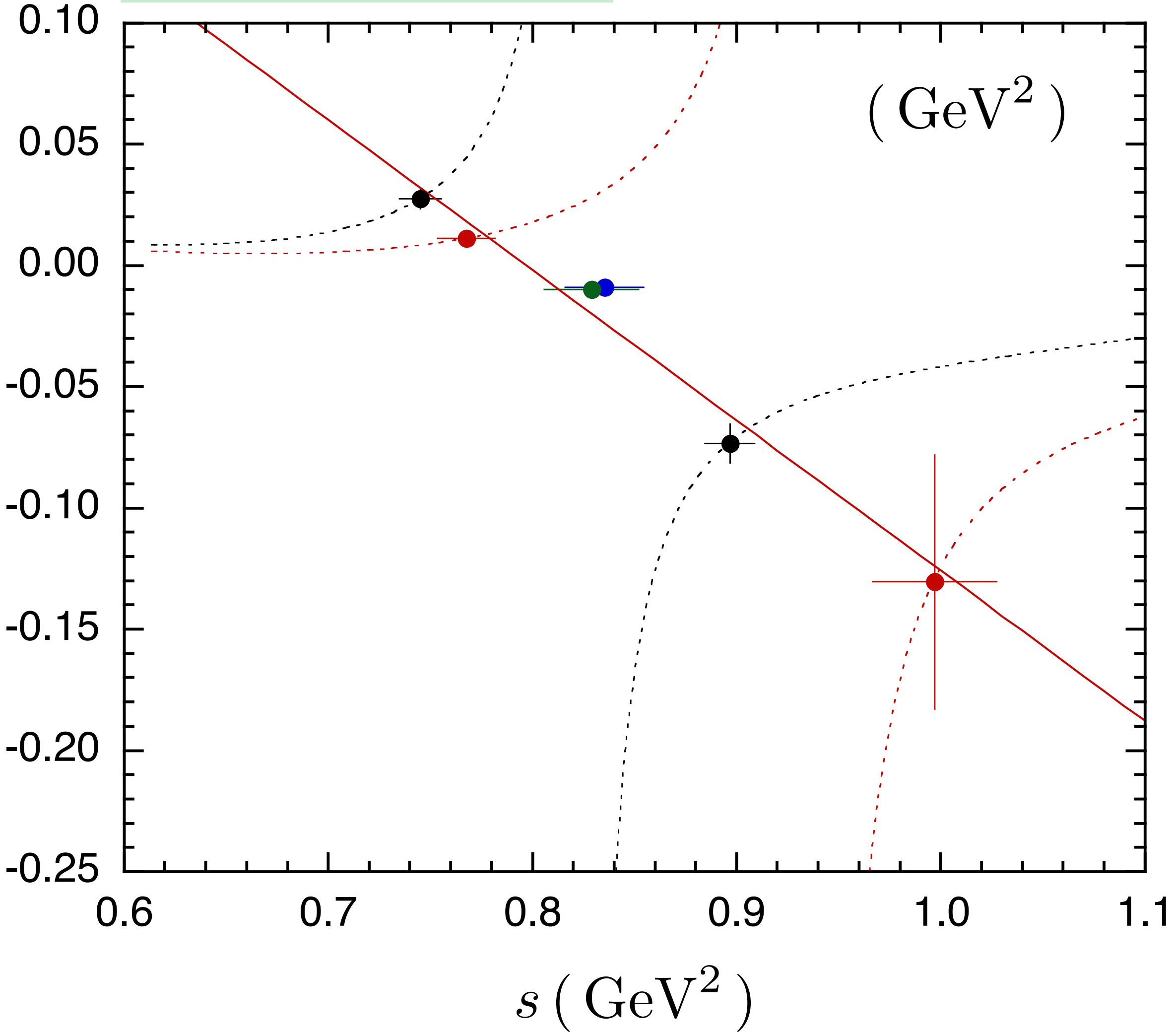
$m_\rho = 775.49(34)\,\mathrm{MeV}$

$N_f = 2+1\,(\mathrm{Wilson}) \quad a = 0.091\,\mathrm{fm}$

$L = 2.9\ \mathrm{fm}\ ,\ m_\pi = 300\,,\ 410\ \mathrm{MeV}$

$$\frac{k^3}{\tan\delta(k)}/\sqrt{s} = \frac{6\pi}{g^2_{\rho\pi\pi}}\cdot(m_\rho^2 - s)$$

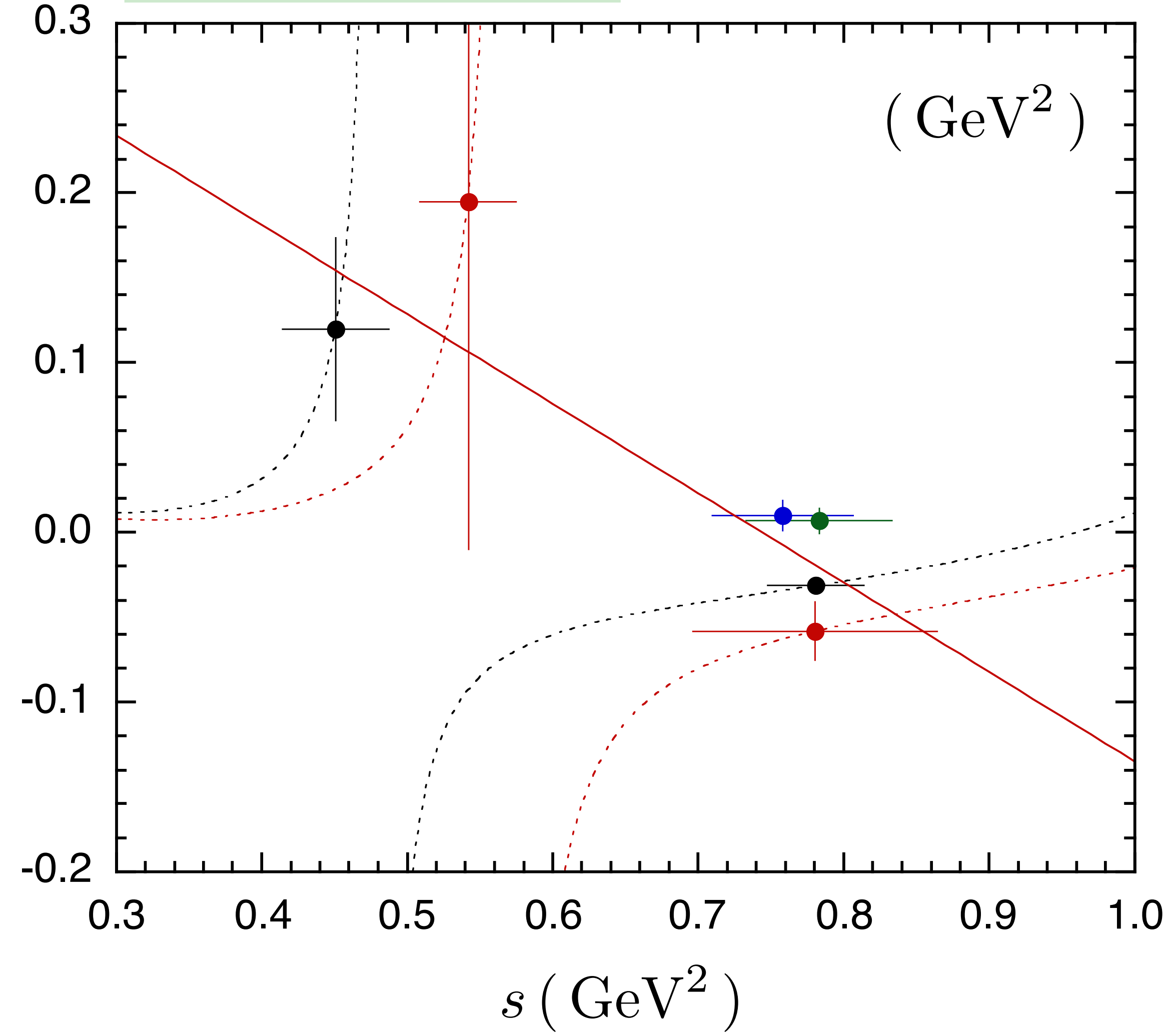
$m_\pi = 410\,\mathrm{MeV}$



$g_{\rho\pi\pi} = 5.51 \pm 0.40$

$m_\rho = 892.8 \pm 5.5 \pm 13\,\mathrm{MeV}$

$m_\pi = 300\,\mathrm{MeV}$



$g_{\rho\pi\pi} = 5.98 \pm 0.56$

$m_\rho = 863 \pm 23 \pm 12\,\mathrm{MeV}$

3. Comparison with other groups

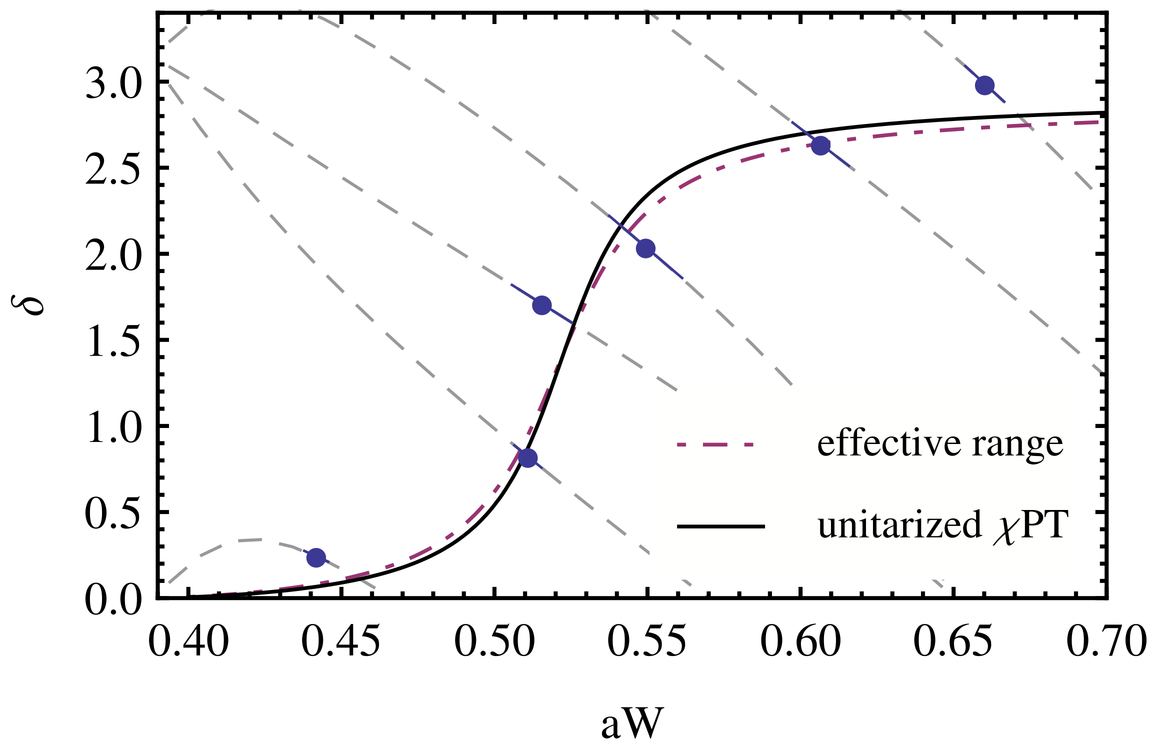
9) Pelissier et.al.

PRD87(2013)014503.

$N_f = 2\,(\text{Wilson}) \quad a = 0.1255\,\text{fm}$

$m_\pi = 300\,\text{MeV}$

$V = 24^2 \times \eta 24 \times 48 \;(\; \eta = 1.0,\ 1.25,\ 2.0\,)$



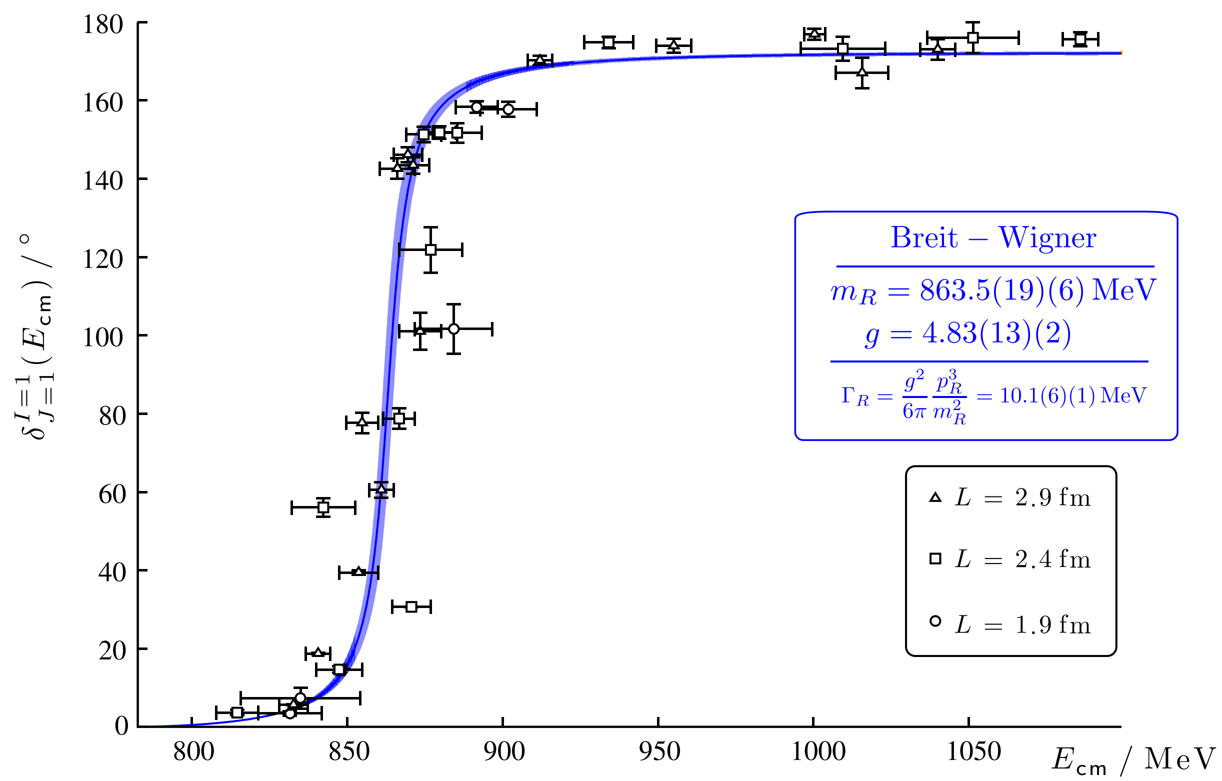
$$g_{\rho\pi\pi} = 6.67(42)$$
$$m_\rho = 827(3)(5)\,\text{MeV}$$
$$\text{at}\;\; m_\pi = 300\,\text{MeV}$$

10) HSC Collaboration

PRD87(2013)034505.

$N_f = 2+1\,(\text{Wilson}) \quad a = 0.12\,\text{fm}$

$L = 1.9, 2.4, 2.9\ \text{fm}\ ,\ m_\pi = 391\ \text{MeV}$

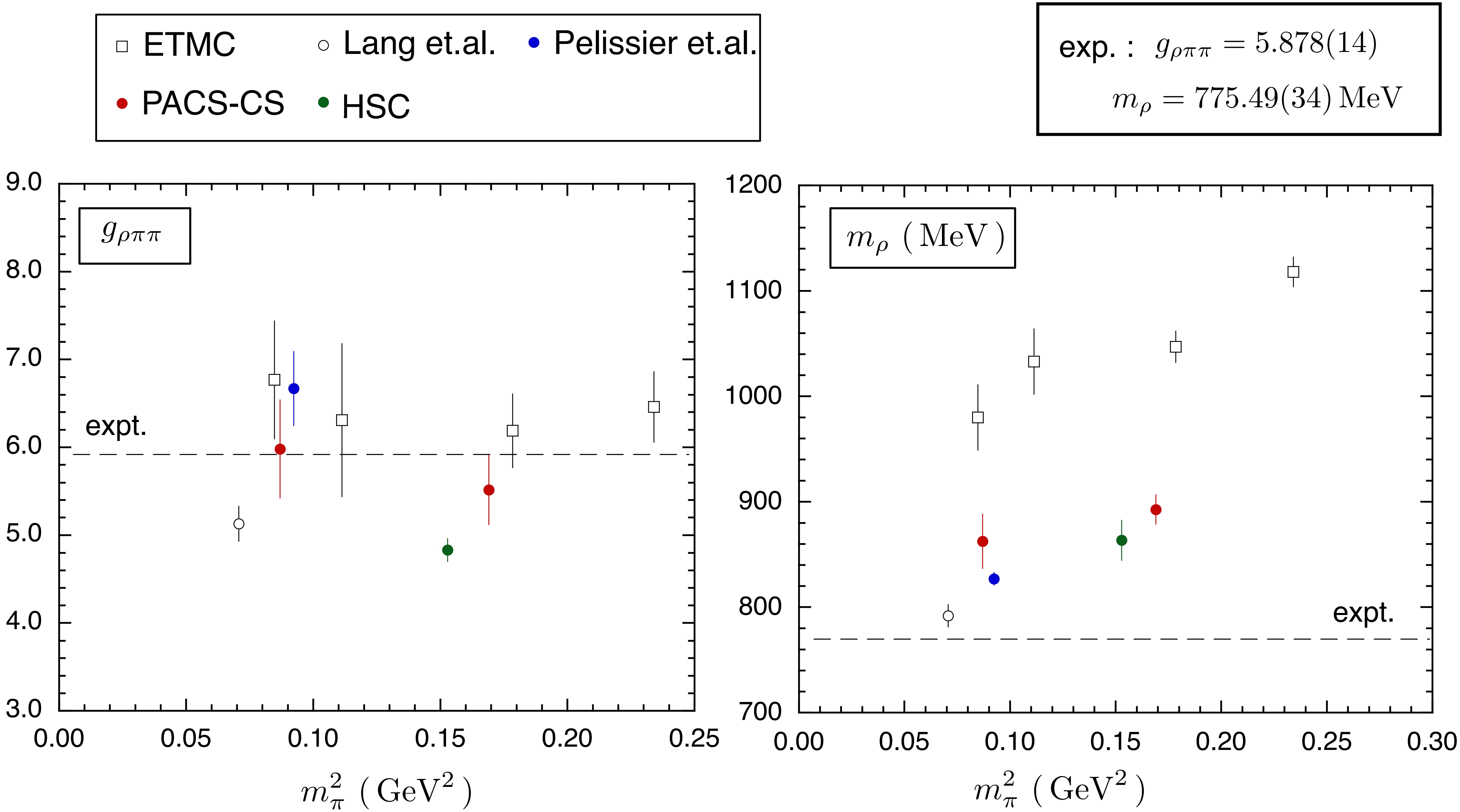


$$g_{\rho\pi\pi} = 4.83(13)(2)$$
$$m_\rho = 863.5(19)(6)\,\text{MeV}$$
$$\text{at}\;\; m_\pi = 391\,\text{MeV}$$

Comparison

$$\text{exp.}: \ g_{\rho\pi\pi} = 5.878(14)$$

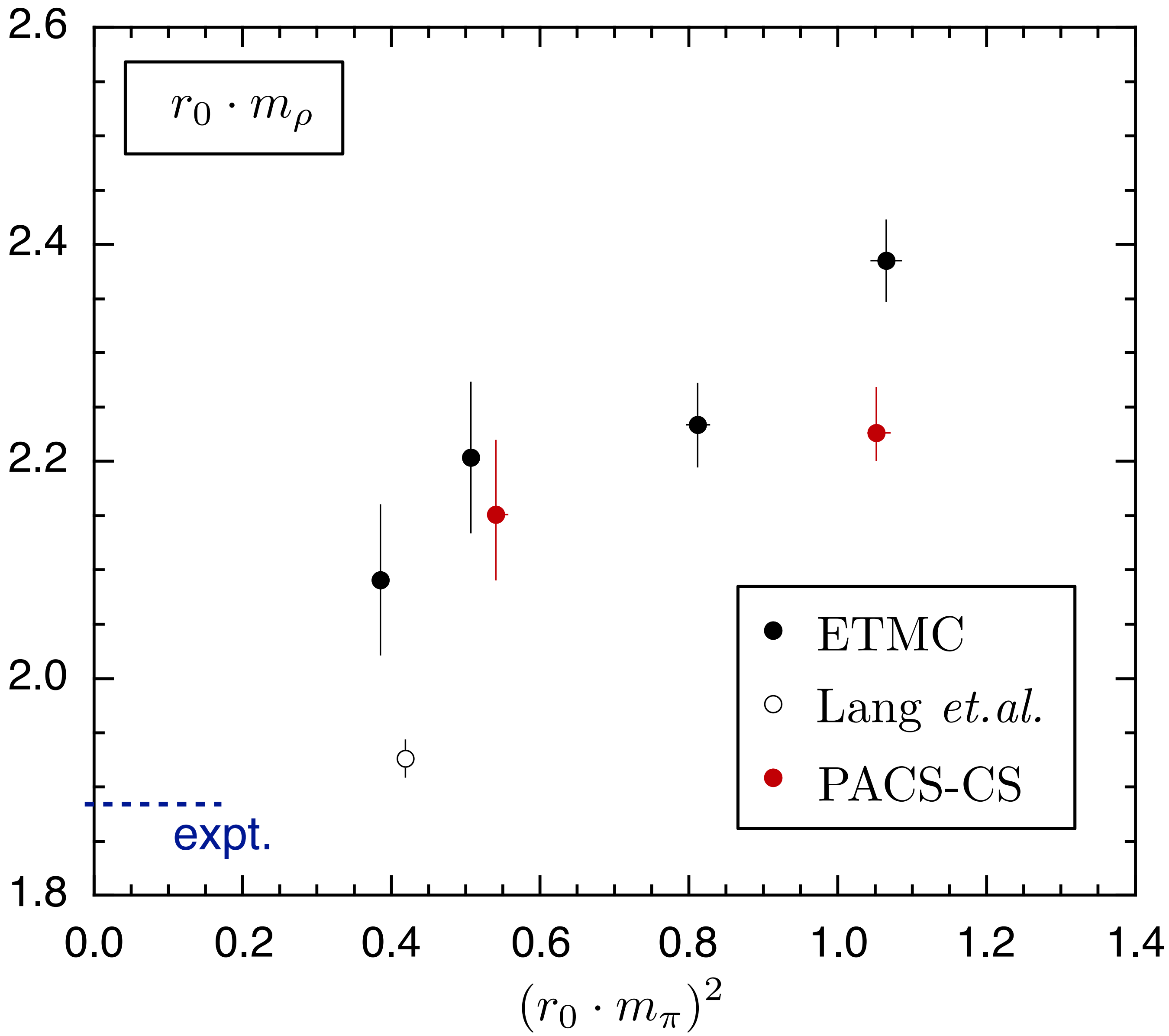
$$m_\rho = 775.49(34)\ \text{MeV}$$



We find discrepancies.

Precise calc. near physical quark mass and cont. limit needs !!

Comparison for dim. less values



Discrepancies still remain.

4. Summary

We calculate SC. phase shift for $I=1$ two-pion system.
We evaluate coupling constant
and resonance energy from it.

exp. :

$$g_{\rho\pi\pi} = 5.878(22)$$
$$m_\rho = 775.5(0.4)\,\mathrm{MeV}$$
$$\Gamma = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{k_\rho^3}{m_\rho^2} = 146.4(1.1)\,\mathrm{MeV}$$

Our results :

- $N_f = 2+1\,(\mathrm{Wilson})$ $a = 0.091\,\mathrm{fm}$
  $L = 2.9$ fm , $m_\pi = 410$ MeV

$$g_{\rho\pi\pi} = 5.65 \pm 0.44$$
$$m_\rho = 892.7 \pm 5.2\,\mathrm{MeV}$$
$$\Gamma_\rho = 135 \pm 21\,\mathrm{MeV}$$
(assuming : $g_{\rho\pi\pi}$: const.)

- $N_f = 2+1\,(\mathrm{Wilson})$ $a = 0.091\,\mathrm{fm}$
  $L = 2.9$ fm , $m_\pi = 300$ MeV

$$g_{\rho\pi\pi} = 5.95 \pm 0.57$$
$$m_\rho = 860 \pm 18\,\mathrm{MeV}$$
$$\Gamma_\rho = 150 \pm 28\,\mathrm{MeV}$$
(assuming : $g_{\rho\pi\pi}$: const.)

We find :

Discrepancies among lattice calculations
for both resonance mass and effective constant.

Precise calc. near physical quark mass and cont. limit needs !!

In Feature works :

- We have to study at phys. point.

  To set $\sqrt{s} \sim m_\rho$

  for $m_\pi \to 0$

  $\pi(\mathbf{p})\pi(-\mathbf{p})$ with $\mathbf{p}$: larger

  Stat. Error. $\to$ large

  1. We have to improve the operator of π with high momentum.
  2. How extract energies of high exicited state ?
  3. Other methods ?

- We have to study other resonances.

  K^*, Δ ....

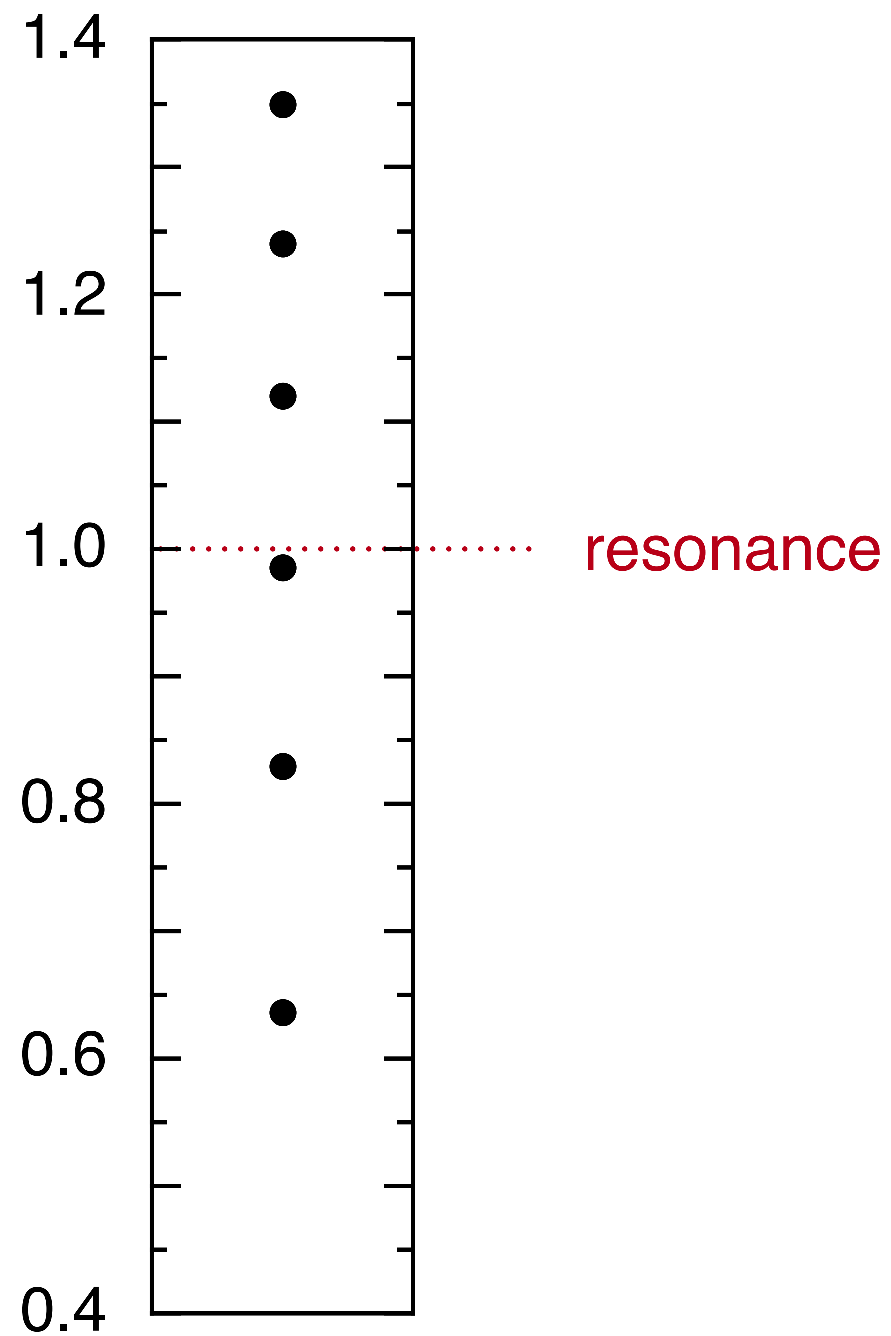
  a_0, κ, σ ....

  charm resonance (X , Y , Z ...)

$\sqrt{s}/m_\rho$ of $\pi\pi$ at physical point

for $\mathbf{P}_{\rm tot} = 0$

on $L = 6\,\mathrm{fm}$



many works remain !!