# QCD at finite temperature and density

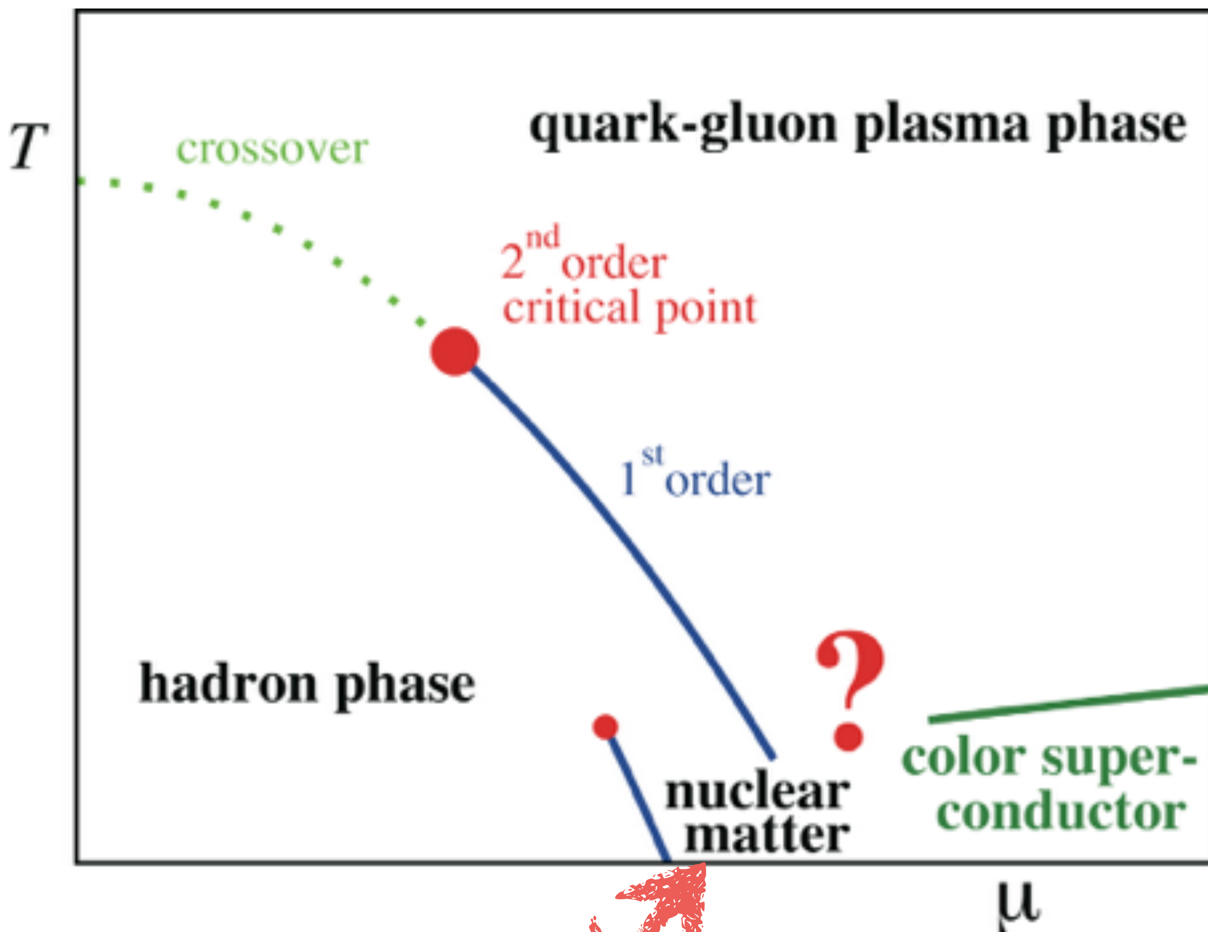
Kazuyuki Kanaya, for the WHOT-QCD Collaboration

WHOT-QCD Collaboration:
S. Aoki, S. Ejiri, T. Hatsuda, N. Ishii, KK, Y. Maezawa, Y. Nakagawa, H. Ohno,
K. Okuno, H. Saito, N. Ukita, T. Umeda, S. Yoshida, …

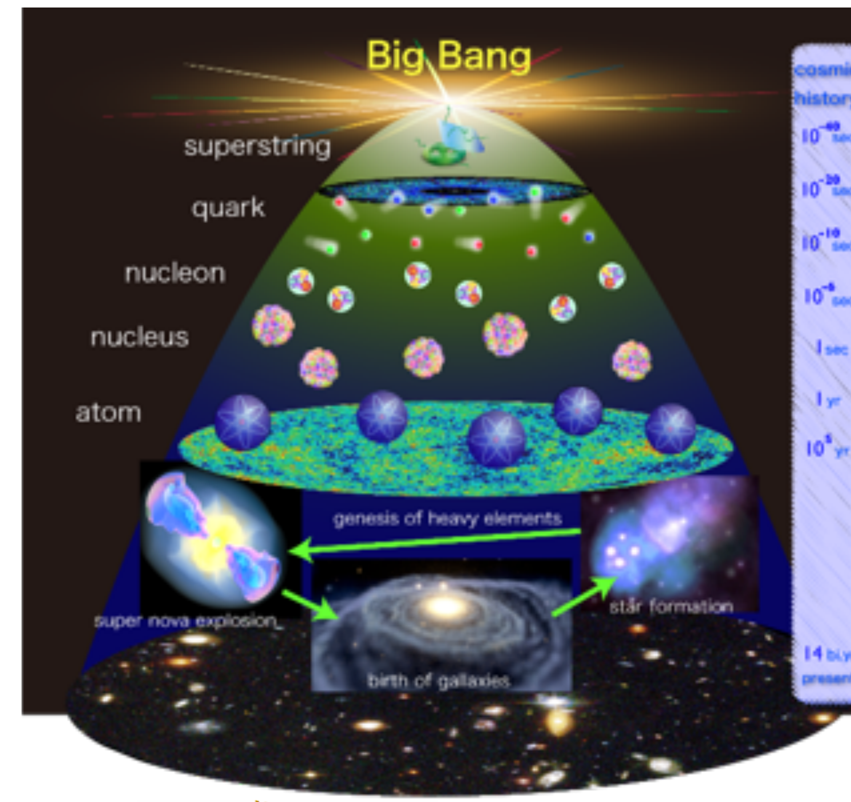
CCS
2014/2/19

# Phase structure of QCD

QCD expected to have a rich phase structure at finite T 's and μ's.




early evolution
of the Universe,
genesis of the
matter, ⋯



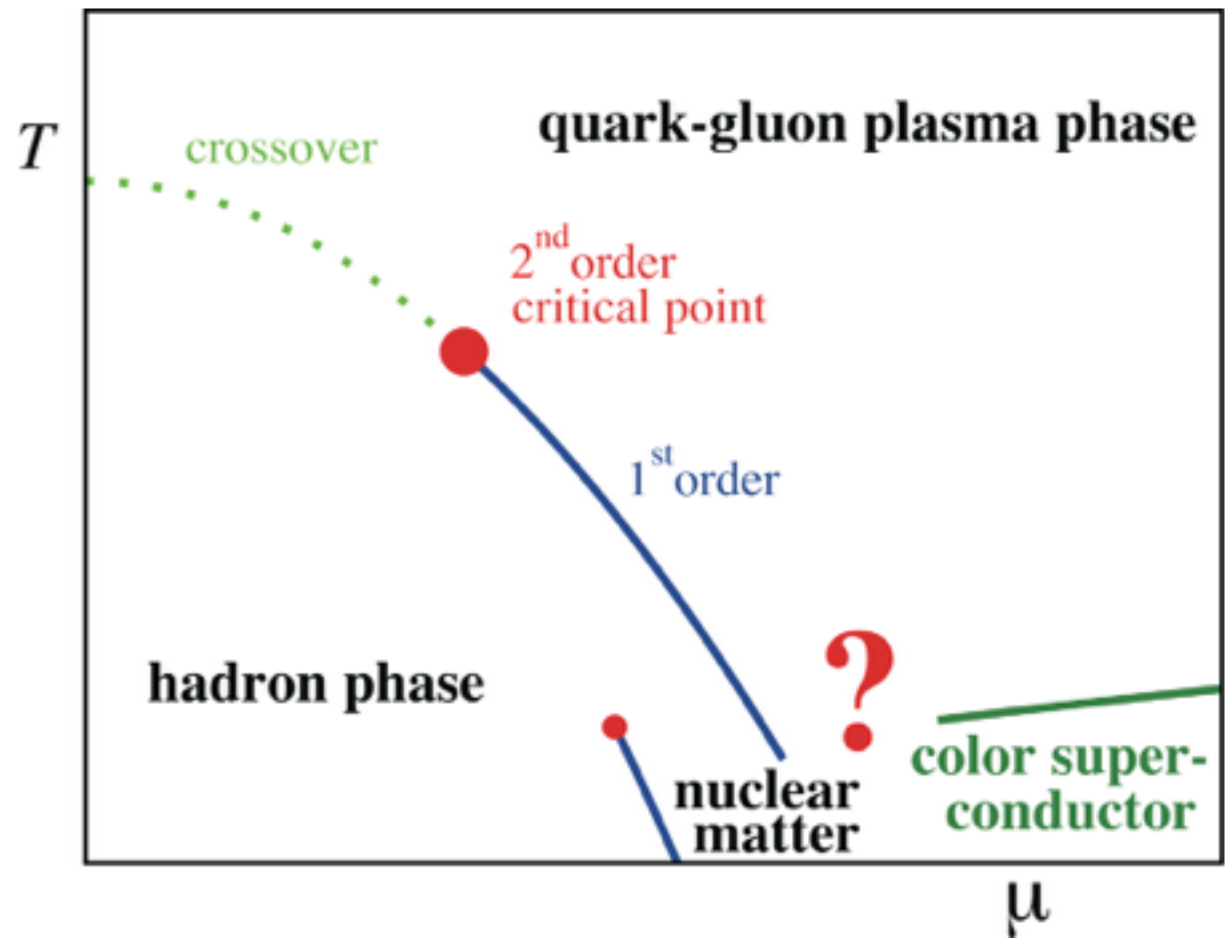

high en. heavy
ion experiments
at RHIC/LHC/...

inner structure of
neutron and quark stars


<= Theoretical inputs directly from the 1st principles of QCD important.

# Our motivations

Theoretical inputs directly from the 1st principles of QCD important.




- properties of the matter in each phase:
  screening length, susceptibilities, …
- precise
  lines / critical points / ...

We study them using improved **Wilson quarks**, which are guaranteed to have the correct continuum limit.

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# Our main results since 2008

- EOS at μ≠0 with dynamical Wilson quarks

  Phys. Rev. D 82 (2010) ref.014508, "Equation of State and Heavy-Quark Free Energy at Finite Temperature and Density in Two Flavor Lattice QCD with Wilson Quark Action ", S. Ejiri, Y. Maezawa, N. Ukita, S. Aoki, T. Hatsuda, N. Ishii, K. Kanaya, T. Umeda

- EOS with fixed-scale approach

  Phys. Rev. D 79 (2009) ref.051501(R), "Fixed Scale Approach to Equation of State in Lattice QCD", T. Umeda, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno

  Phys. Rev. D 85 (2012) ref.094508, "Equation of state in 2+1 flavor QCD with improved Wilson quarks by the fixed scale approach", T. Umeda, S. Aoki, S. Ejiri, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno

- Search for the critical point with histogram method

  Phys. Rev. D 84 (2011) ref.054502, "Phase structure of finite temperature QCD in the heavy quark region", H. Saito, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno, T. Umeda

  Phys. Rev. D (2014) in press, "Histograms in heavy-quark QCD at finite temperature and density", H. Saito, S. Ejiri, S. Aoki, K. Kanaya, Y. Nakagawa, H. Ohno, K. Okuno, T. Umeda

- Q-$\overset{(-)}{\mathrm{Q}}$ interaction and screening masses at finite T and μ

  Phys. Rev. D 81 (2010) ref.091501(R), "Electric and Magnetic Screening Masses at Finite Temperature from Generalized Polyakov-Line Correlations in Two-flavor Lattice QCD", Y. Maezawa, S. Aoki, S. Ejiri, T. Hatsuda, N. Ishii, K. Kanaya, N. Ukita and T. Umeda

  Prog. Theor. Phys. 128 (2012), "Application of fixed scale approach to static quark free energies in quenched and 2 + 1 flavor lattice QCD with improved Wilson quark action", Y. Maezawa, T. Umeda, S. Aoki, S. Ejiri, T. Hatsuda, K. Kanaya and H. Ohno

- Charmonium dissociation with variational method

  Phys. Rev. D 84 (2011) ref.094504, "Charmonium spectral functions with the variational method in zero and finite temperature lattice QCD", H. Ohno, S. Aoki, S. Ejiri, K. Kanaya, Y. Maezawa, H. Saito and T. Umeda

# Lattice QCD at μ≠0

- LQCD at $\mu \neq 0$ $\quad U_4 \longrightarrow \begin{cases} U_4 e^{\mu a} & \cdots \quad \text{positive } t \text{ direction} \\ U_4 e^{-\mu a} & \cdots \quad \text{nagative } t \text{ direction} \end{cases}$

  => $[\det M(\mu)]^* = \det M(-\mu^*) \neq \det M(\mu)$

  => MC based on importance sampling with det$M$ not justified

- Sign problem (complex phase problem)

  phase-quenched simulation by det $M$ –> |det $M$|, and handling the phase in the measurement

"reweighting" $\langle \mathcal{O} \rangle = \dfrac{\int d\Phi\, \mathcal{O}\, e^{-S}}{\int d\Phi\, e^{-S}}$

$$= \frac{\int d\Phi\, \mathcal{O} e^{-\Delta S}\, e^{-S+\Delta S}}{\int d\Phi\, e^{-\Delta S}\, e^{-S+\Delta S}} = \frac{\langle \mathcal{O} e^{-\Delta S} \rangle_{S-\Delta S}}{\langle e^{-\Delta S} \rangle_{S-\Delta S}}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int DU\, \mathcal{O}\, [\det M]^{N_F} e^{-S_g} = \frac{\langle \mathcal{O} e^{iN_F\theta} \rangle_{\text{p.q.}}}{\langle e^{iN_F\theta} \rangle_{\text{p.q.}}}, \qquad \det M = |\det M|\, e^{i\theta}$$

=> Exponentially high statistic required when $\theta$ fluctuates a lot (<= large $\mu$).

# Lattice QCD at μ≠0

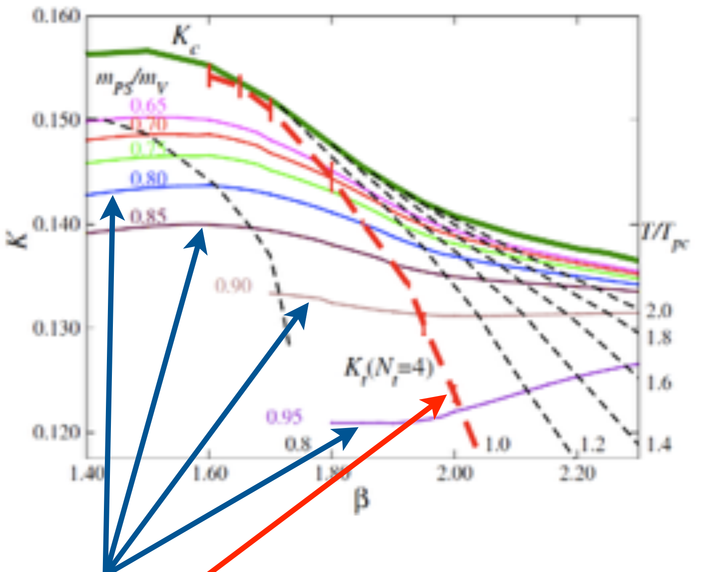
- Techniques for small $\mu/T$
  - ✦ Taylor expansion around $\mu = 0$
  - ✦ multi-parameter reweighting
  - ✦ imaginary $\mu$ (analytic continuation to real $\mu$)
  - ✦ canonical ensemble
  - ✦ complex Langevin
  - ✦ Lefschetz thimble etc. etc.
  - ✦ combination of them & other techniques to extend the range of applicability

# EOS at μ≠0 with Wilson quarks

Our previous study at μ=0: $N_F = 2$ QCD, $Nt$=4, (6)

PRD 63, 034502 (2001);
PRD 64, 074510 (2001);
PRD 75, 074501 (2007)

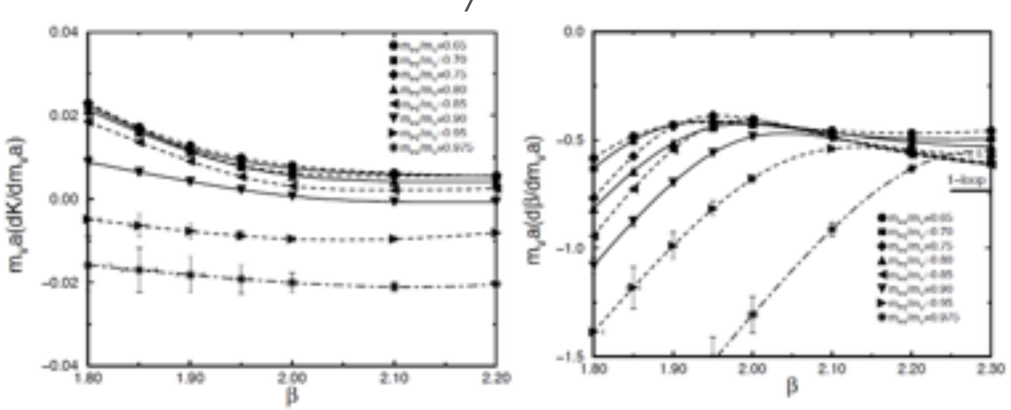
## Lines of constant physics




LCP's defined by $m_\pi/m_\rho$ at $T = 0$

Finite $T$ crossover for $Nt$=4.

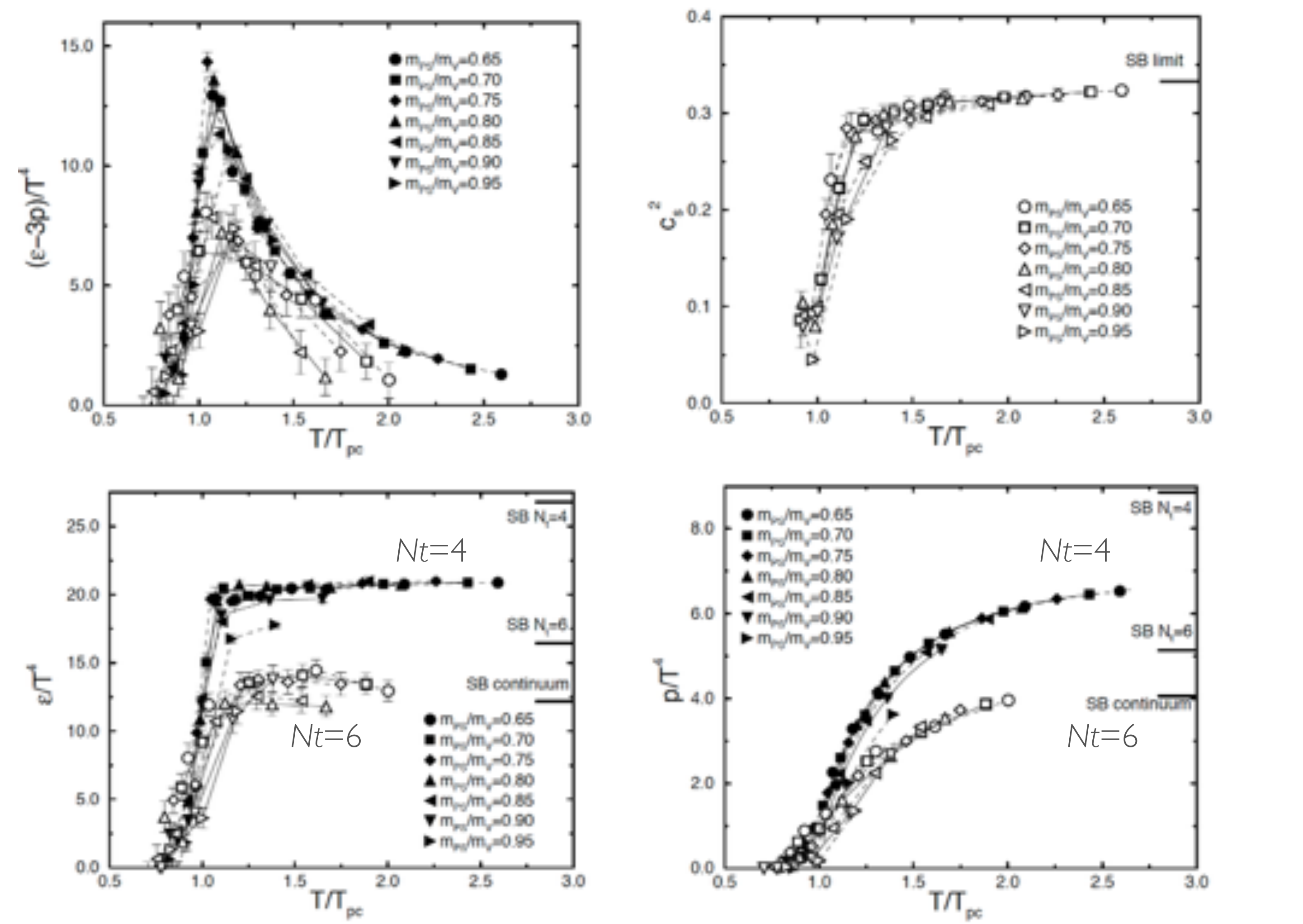
## Beta functions by inverse matrix method




## Equation of state by integration method

$$\frac{\epsilon - 3p}{T^4} = N_t^4 \left\{ \left\langle a\frac{dS}{da} \right\rangle - \left\langle a\frac{dS}{da} \right\rangle_{T=0} \right\} = \frac{N_t^3}{N_s^3} \sum_i a \frac{db_i}{da} \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}$$

$$p = \frac{T}{V}\int_{b_0}^{b} db \frac{1}{Z}\frac{\partial Z}{\partial b} = -\frac{T}{V}\int_{b_0}^{b} \sum_i db_i \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}$$



# EOS at μ≠0 with Wilson quarks

## μ≠0 by the Taylor expansion method:

- Observables

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z} \equiv \omega, \qquad \frac{n_f}{T^3} = \frac{1}{VT^3} \frac{\partial \ln \mathcal{Z}}{\partial(\mu_f/T)} = \frac{\partial(p/T^4)}{\partial(\mu_f/T)}, \qquad (f = u,\, d)$$

$$\frac{\chi_q}{T^2} = \left( \frac{\partial}{\partial(\mu_u/T)} + \frac{\partial}{\partial(\mu_d/T)} \right) \frac{n_u + n_d}{T^3} \qquad \frac{\chi_I}{T^2} = \left( \frac{\partial}{\partial(\mu_u/T)} - \frac{\partial}{\partial(\mu_d/T)} \right) \frac{n_u - n_d}{T^3}$$

- Taylor expansion in $\mu_u=\mu_d(=\mu_q)$ at $\mu_u=\mu_d=0$

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n, \qquad c_n(T) = \frac{1}{n!} \frac{N_t^3}{N_s^3} \left. \frac{\partial^n \ln \mathcal{Z}}{\partial(\mu_q/T)^n} \right|_{\mu_q=0}$$

$$\frac{\chi_q(T,\mu_q)}{T^2} = 2c_2 + 12c_4 \left(\frac{\mu_q}{T}\right)^2 + \cdots \qquad c_2 = \frac{N_t}{2N_s^3}\mathcal{A}_2, \quad c_4 = \frac{1}{4!N_s^3 N_t}(\mathcal{A}_4 - 3\mathcal{A}_2^2)$$

$$\mathcal{A}_2 = \langle \mathcal{D}_2 \rangle + \langle \mathcal{D}_1^2 \rangle, \qquad \mathcal{A}_4 = \langle \mathcal{D}_4 \rangle + 4\langle \mathcal{D}_3 \mathcal{D}_1 \rangle + 3\langle \mathcal{D}_2^2 \rangle + 6\langle \mathcal{D}_2 \mathcal{D}_1^2 \rangle + \langle \mathcal{D}_1^4 \rangle$$

$$\mathcal{D}_n = N_f \frac{\partial^n \ln \det M}{\partial \mu^n} \qquad \mathcal{D}_1 = N_f \mathrm{tr}\left( M^{-1} \frac{\partial M}{\partial \mu} \right)$$

$$\mu \equiv \mu_q a. \qquad \mathcal{D}_2 = N_f \left[ \mathrm{tr}\left( M^{-1} \frac{\partial^2 M}{\partial \mu^2} \right) - \mathrm{tr}\left( M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} \right) \right]$$

Phys. Rev. D 82 (2010) ref.014508, “Equation of State and Heavy-Quark Free Energy at Finite Temperature and Density in Two Flavor Lattice QCD with Wilson Quark Action “, S. Ejiri, Y. Maezawa, N. Ukita, S. Aoki, T. Hatsuda, N. Ishii, K. Kanaya, T. Umeda

# EOS at μ≠0 with Wilson quarks

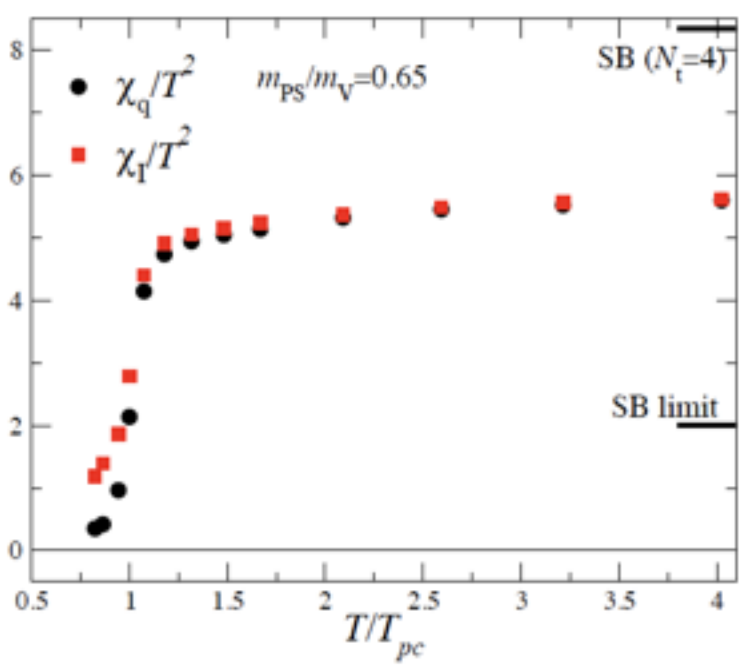
- Noise method for traces in the Taylor coefficients

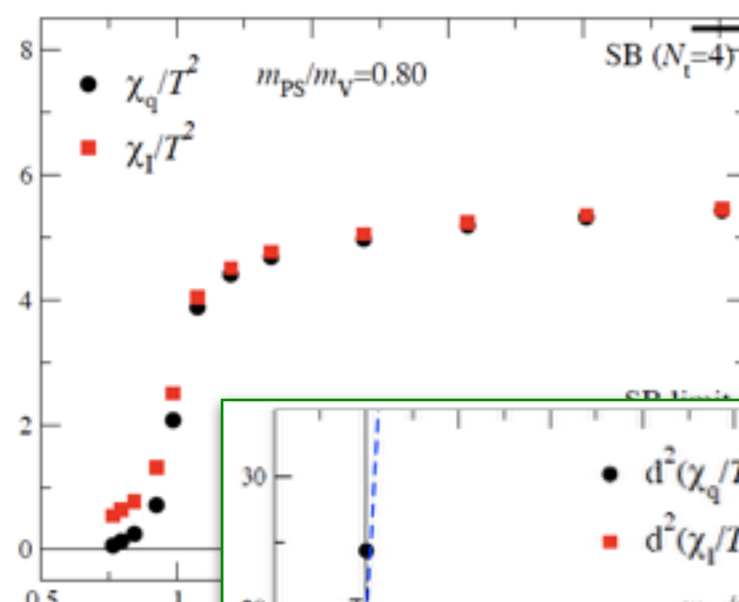
$$\frac{1}{N_{\rm noise}}\sum_i \eta_{i,\alpha}\eta^*_{i,\beta} \approx \delta_{\alpha,\beta} \qquad {\rm tr}\left(\frac{\partial^n M}{\partial^n \mu}M^{-1}\right) \approx \frac{1}{N_{\rm noise}}\sum_i \eta_i^\dagger \frac{\partial^n M}{\partial^n \mu}X_i; \quad X_i = M^{-1}\eta_i$$

For Wilson quarks, we generate independent η for each color and spin.
Error from $D_1$ turned out to be dominating in the results. => about 100 times larger $N_{\rm noise}$ for $D_1$.

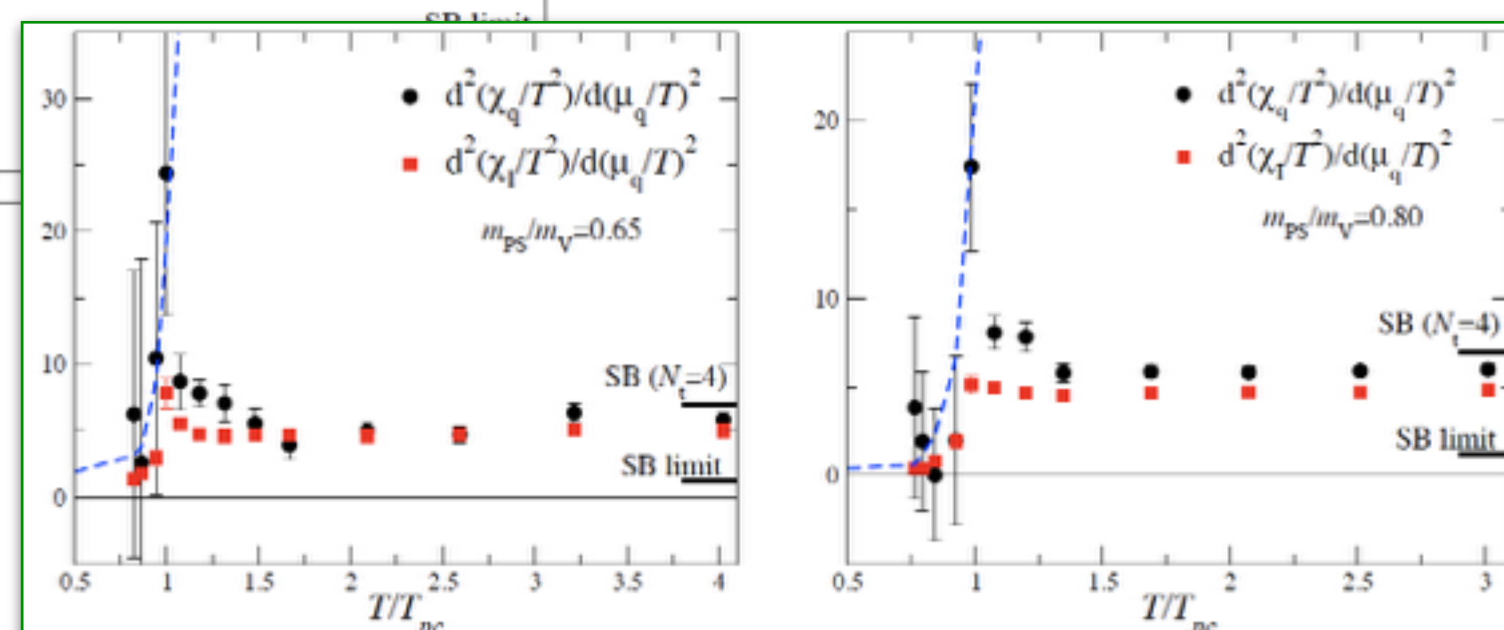
- Quark number susceptibilities at μ=0

$N_F$ = 2 QCD, $Nt$=4




- Lowest order Taylor coeff.



# EOS at μ≠0 with Wilson quarks

- $\chi_q$

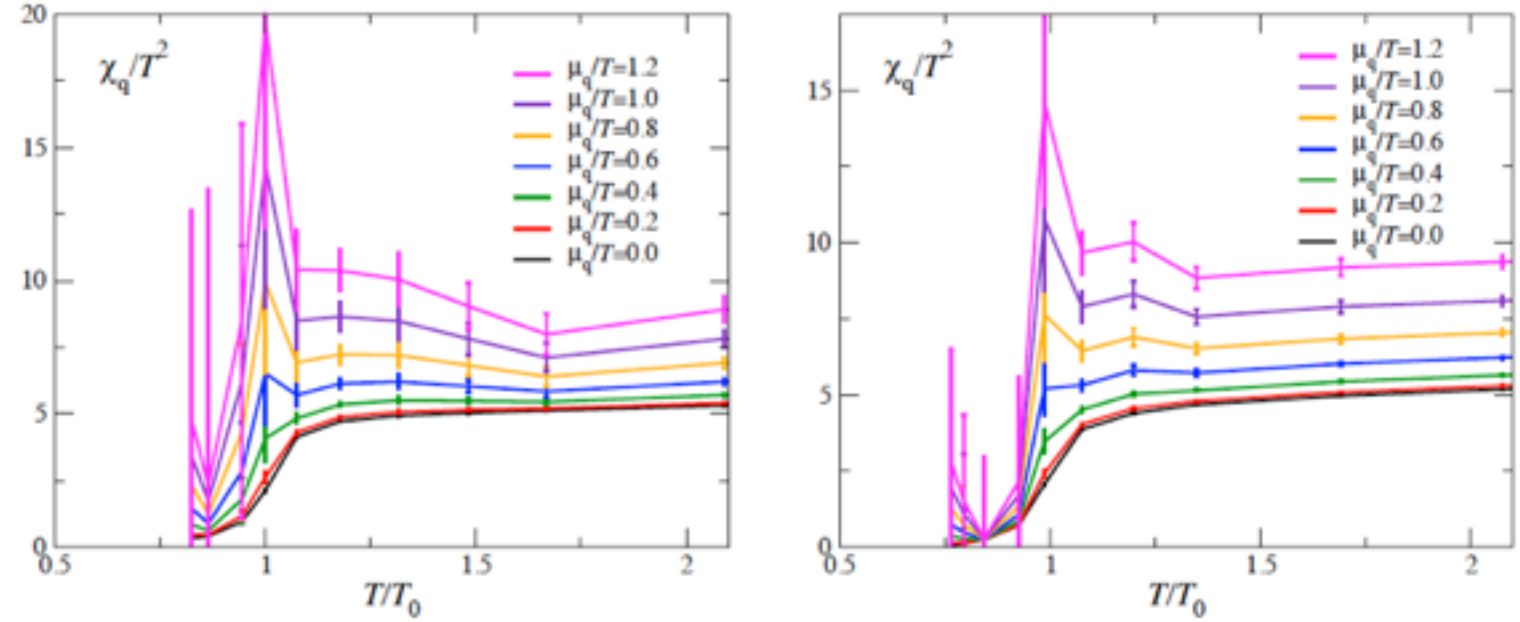



FIG. 14: Quark number susceptibility at finite $\mu_q$ for $m_{\rm PS}/m_{\rm V} = 0.65$ (left) and 0.80 (right).

- $\chi_I$

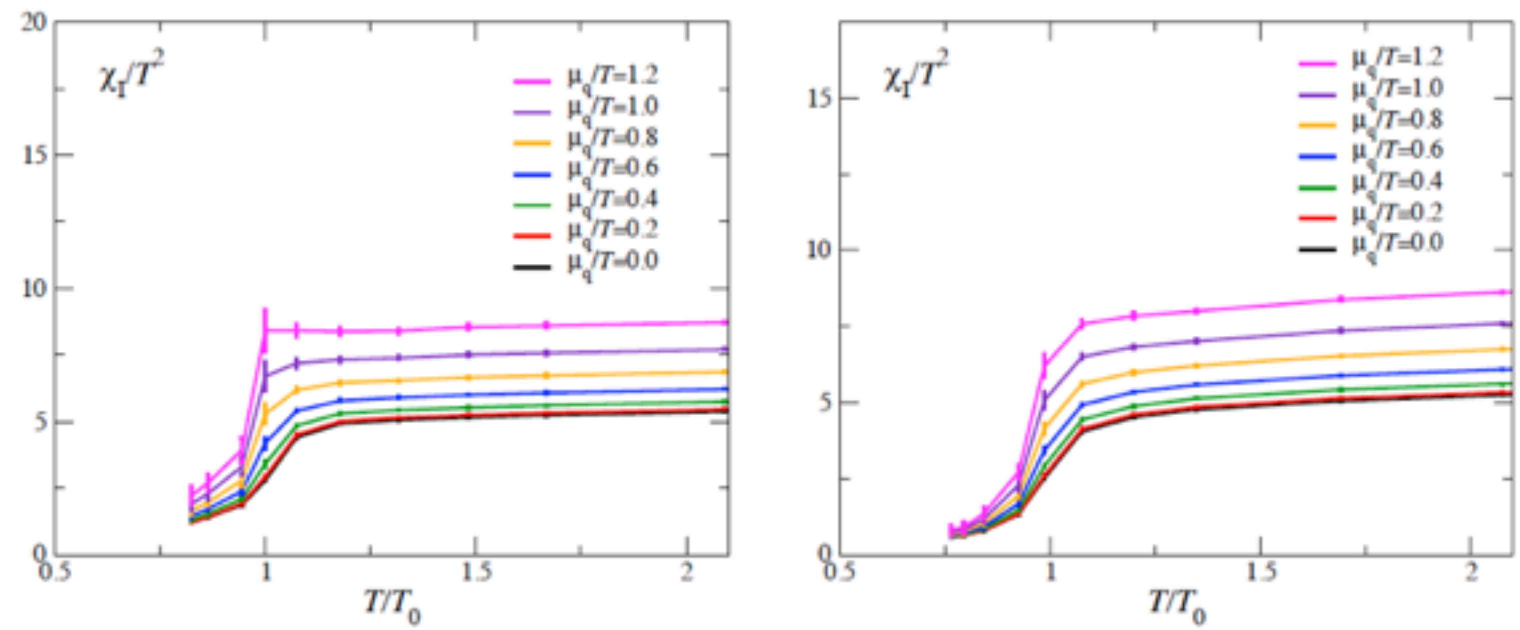



FIG. 15: Isospin susceptibility at finite $\mu_q$ for $m_{\rm PS}/m_{\rm V} = 0.65$ (left) and 0.80 (right).

**Suggest critical pt. at finite μ**, which is insensitive to the iso-spin number.

# EOS at μ≠0 with Wilson quarks

To further improve the calculation

Allton et al., PRD 66, 074507 ('02)
Ejiri, PRD 77, 014508 ('08)

## ▸A hybrid Taylor+reweighting method

Reweight the grand canonical partition function from μ=0:

$$\mathcal{Z}(T,\mu_q) = \mathcal{Z}(T,0) \left\langle \left( \frac{\det M(\mu)}{\det M(0)} \right)^{N_f} \right\rangle_{(\mu_q=0)} \equiv \mathcal{Z}(T,0) \left\langle e^{F(\mu)} e^{i\theta(\mu)} \right\rangle_{(\mu_q=0)}$$

and Taylor-expand the terms in exp:

$$F(\mu) \equiv N_f \mathrm{Re}\left[ \ln \left( \frac{\det M(\mu)}{\det M(0)} \right) \right] = N_f \sum_{n=1}^{\infty} \frac{1}{(2n)!} \mathrm{Re}\left[ \frac{\partial^{2n} (\ln \det M)}{\partial \mu^{2n}} \right]_{(\mu=0)} \mu^{2n} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \mathrm{Re}\mathcal{D}_{2n} \mu^{2n}$$

$$\theta(\mu) = N_f \mathrm{Im}\left[ \ln \det M(\mu) \right] = N_f \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \mathrm{Im}\left[ \frac{\partial^{2n+1} (\ln \det M(\mu))}{\partial \mu^{2n+1}} \right]_{(\mu=0)} \mu^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \mathrm{Im}\mathcal{D}_{2n+1} \mu^{2n+1}$$

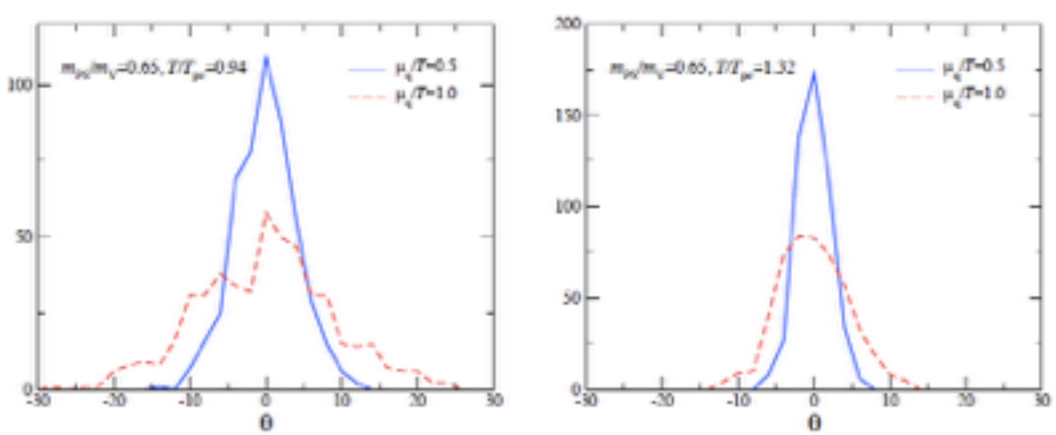
Truncate the expansions up to $D_4$

* Identical to the truncated Taylor expansion up to the 4th order, but contain a part of higher orders through the exponential function.

* Exact for free QGP, in which $D_n=0$ for $n>4$, => The truncation will be OK at high $T$.

## ▸Gaussian approximation for θ distribution at small μ

Ejiri, PRD 77, 014508 ('08)




Sign problem avoided at small μ
=> See discussions later.

# EOS at μ≠0 with Wilson quarks

- $\chi_q$

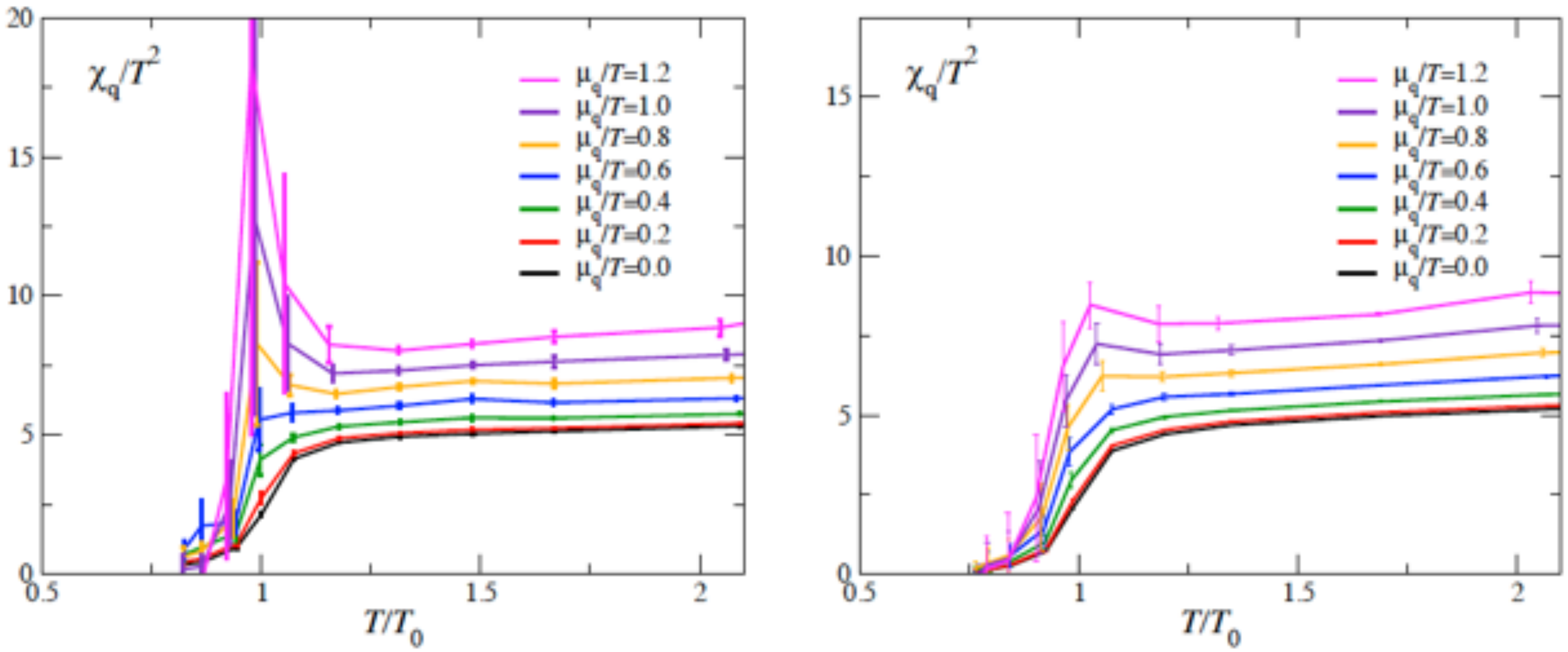
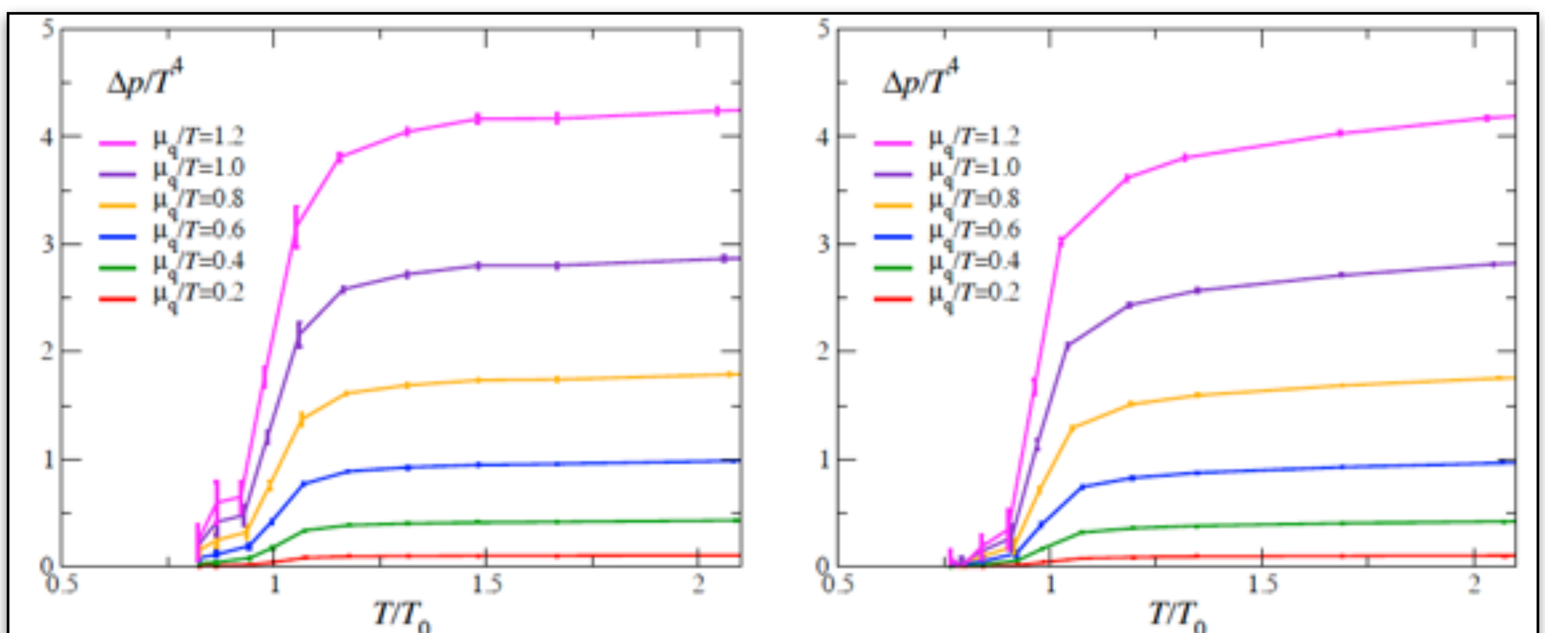



FIG. 24: Quark number susceptibility for each $\mu_q/T$ at $m_{\rm PS}/m_{\rm V} = 0.65$ (left) and 0.80 (right).

The signal became less noisy.

Suggesting more clearly **critical point at finite μ**.

- $p(\mu)$-$p(0)$




The first EOS at μ≠0 with Wilson-type quarks.

# EOS at μ≠0 with Wilson quarks

- heavy quark free energies at μ≠0

$$V^R(r,T,\mu_q) = v_0^R + v_1^R\left(\frac{\mu_q}{T}\right) + v_2^R\left(\frac{\mu_q}{T}\right)^2 + O(\mu^3)$$

$R = \mathbf{1},\ \mathbf{8},\ \mathbf{6},\ \mathbf{3}^*$

$$\begin{aligned}
\Omega^{\mathbf{1}}(r) &= \frac{1}{3}\mathrm{tr}\Omega^\dagger(\mathbf{x})\Omega(\mathbf{y}),\\
\Omega^{\mathbf{8}}(r) &= \frac{1}{8}\mathrm{tr}\Omega^\dagger(\mathbf{x})\mathrm{tr}\Omega(\mathbf{y}) - \frac{1}{24}\mathrm{tr}\Omega^\dagger(\mathbf{x})\Omega(\mathbf{y}),\\
\Omega^{\mathbf{6}}(r) &= \frac{1}{12}\mathrm{tr}\Omega(\mathbf{x})\mathrm{tr}\Omega(\mathbf{y}) + \frac{1}{12}\mathrm{tr}\Omega(\mathbf{x})\Omega(\mathbf{y}),\\
\Omega^{\mathbf{3}^*}(r) &= \frac{1}{6}\mathrm{tr}\Omega(\mathbf{x})\mathrm{tr}\Omega(\mathbf{y}) - \frac{1}{6}\mathrm{tr}\Omega(\mathbf{x})\Omega(\mathbf{y}),
\end{aligned}$$

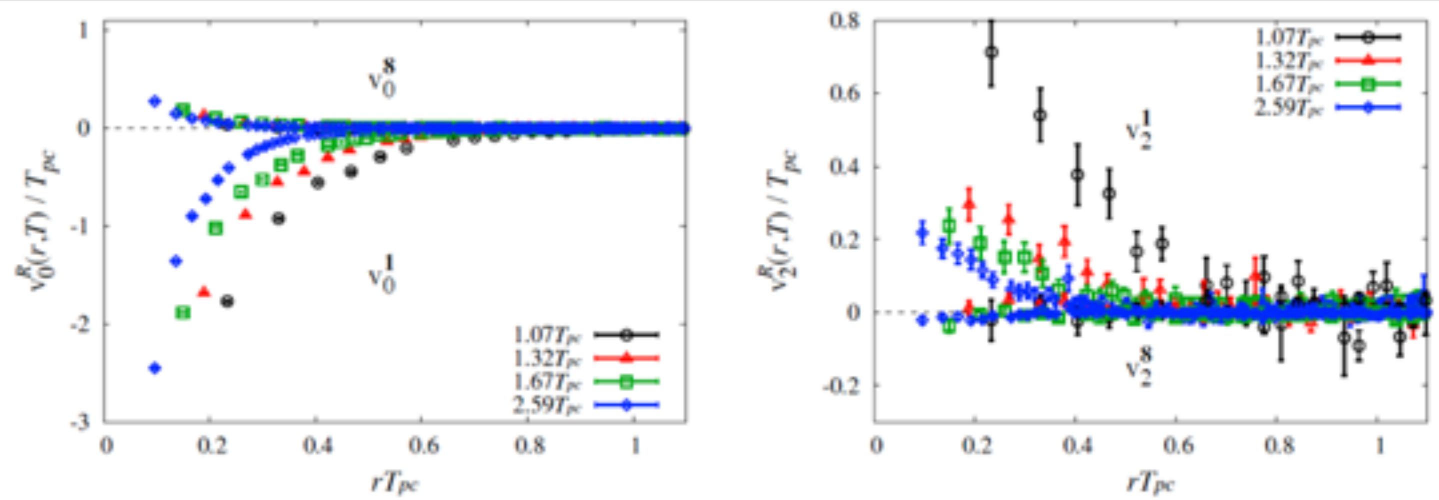



FIG. 25: $v_0^R$ (left) and $v_2^R$ (right) for color-singlet and octet $Q\bar{Q}$ channels above $T_{pc}$ at $m_{PS}/m_V = 0.65$.

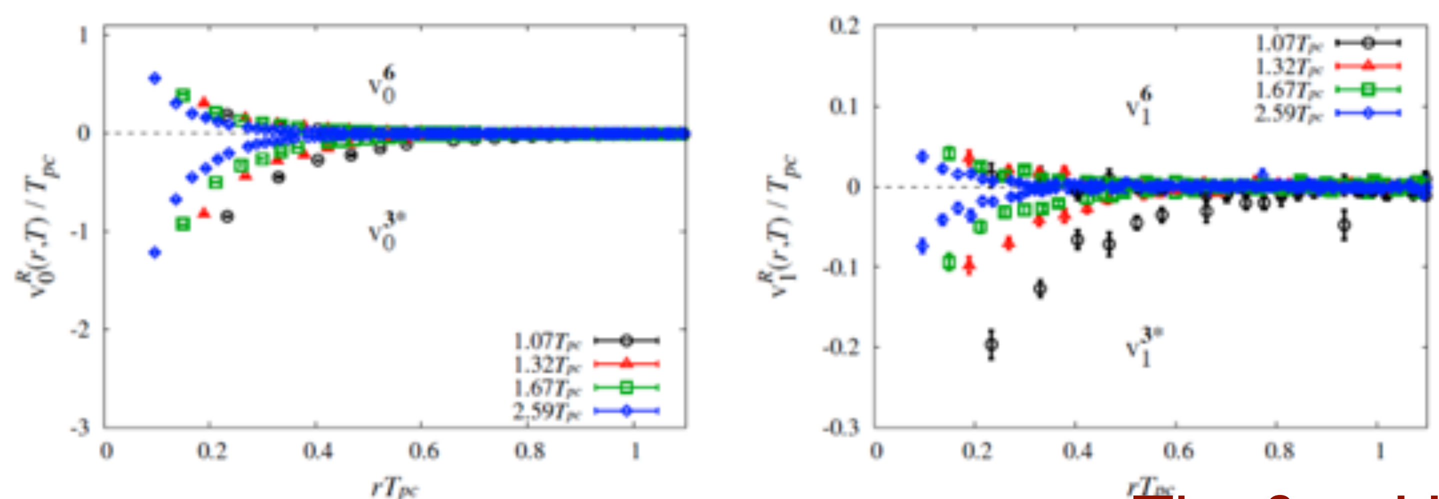



FIG. 26: $v_0^R$ (left) and $v_1^R$ (right) for color-sextet and antitriplet $QQ$ channels above $T_{pc}$ at $m_{PS}/m_V = 0.65$.

$Q\bar{Q}$ interaction:
weaker at μ≠0

QQ interaction:
stronger at μ≠0

(leading order in μ)

The first HQFE at μ≠0 with Wilson-type quarks.

# EOS with fixed-scale approach

We want to extend the studies to $N_F$=2+1, larger $N_t$, and smaller $m_q$

*More improvements / developments of the method needed.*

We note: a large fraction of the cost due to $T = 0$ simulations

- Determination of LCP, scale, beta functions, etc.
- $T = 0$ subtractions for renormalization (needed at all the simulation points!)

↓

**Fixed scale approach** with a **T-integration method**.

Vary $T = \frac{1}{N_t a}$ by varying $N_t$ with all coupling params. fixed.

Conventional integration method not applicable (integration in the coupl. param. space).

$$\frac{p}{T^4} = \int_{T_0}^{T} dT \, \frac{\epsilon - 3p}{T^5} \quad \longleftarrow \quad T \frac{\partial}{\partial T}\left(\frac{p}{T^4}\right) = \frac{\epsilon - 3p}{T^4}$$

Pros and cons: T=0 simulation costs largely removable

- A common T=0 simulation enough for all T=0 subtractions. We can even borrow publicly available configurations on ILDG.
- Automatically on a LCP w/o fine tuning..

Limited resolution in T => need to check if it works on isotropic lattices.

Other properties are complementary to the fixed-Nt approach.

Roughly speaking

| | fixed Nt | fixed scale |
|---|---|---|
| T ≈ Tc | ☹ | ☺ |
| T > (2-3)*Tc | ☺ | ☹ |

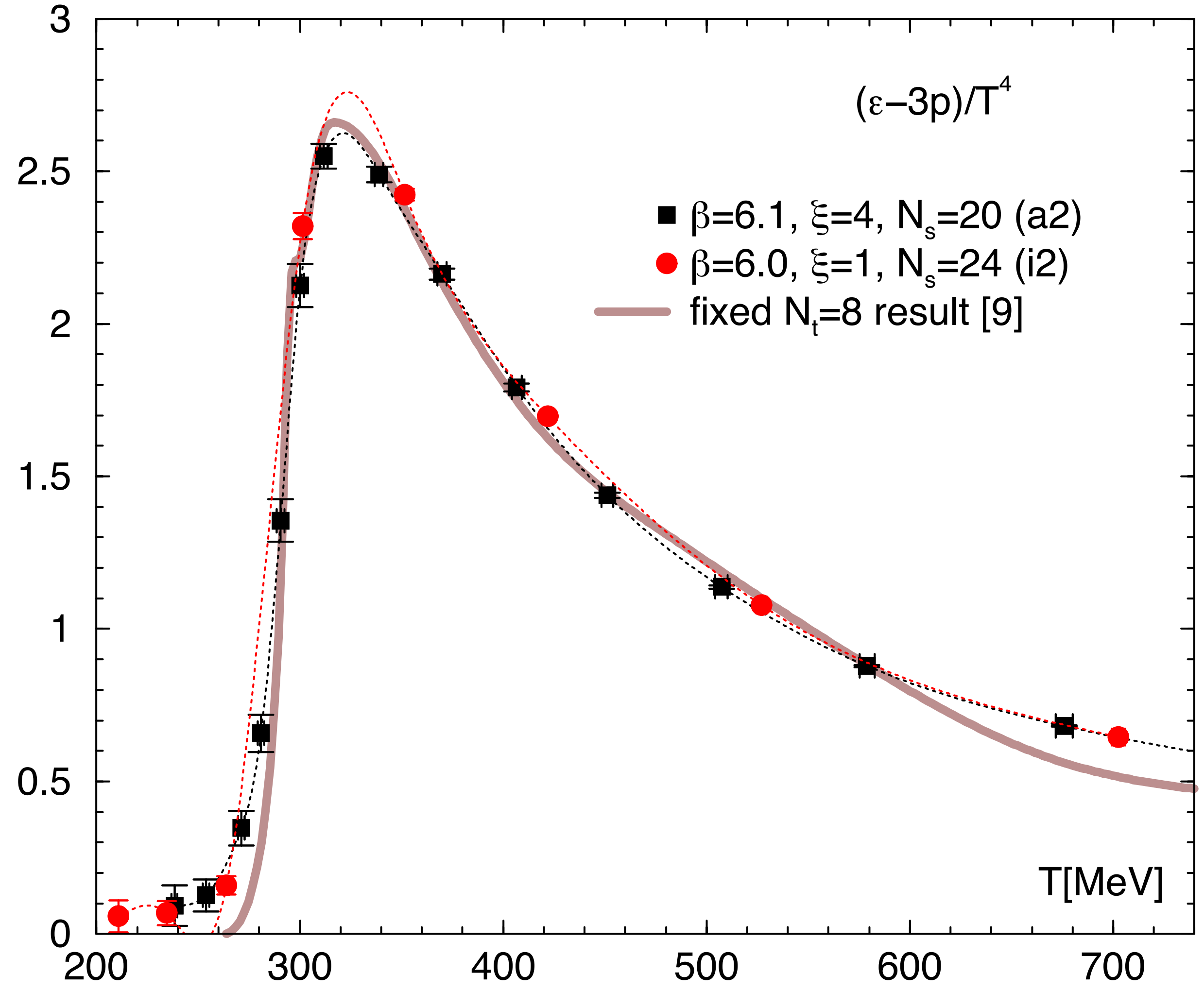
- Keep the lattice spacing small
- More costs due to larger *Nt*.

- *Nt* too small
- Keep the lattice volume large.

# EOS with fixed-scale approach

**A test in quenched QCD** => looks fine.

Phys. Rev. D 79 (2009) ref.051501(R), "Fixed Scale Approach to Equation of State in Lattice QCD", T. Umeda, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno




Results compared among

- ★ **fixed-scale on isotropic lattice**
  (*as*~0.095fm, *Nt*=3-10 => *T*=200-700MeV, *Ls*~1.5fm)
- ★ **fixed-scale on anisotropic lattice**
  (ξ=4, i.e. 4-times smaller *at* => 4-times finer *T*-resolution)
- ★ **fixed-Nt approach**
  (*Nt*=8 by Boyd el al. NPB469(96): *Ns*=32 => *Ls*~2.7fm around *Tc*)

Note: effects due to small *Ls* are physical finite volume effects, i.e not due to the algorithm.

Besides understandable deviations, results consistent with each other

➡ **T-interpolation under control on the isotropic lattice**

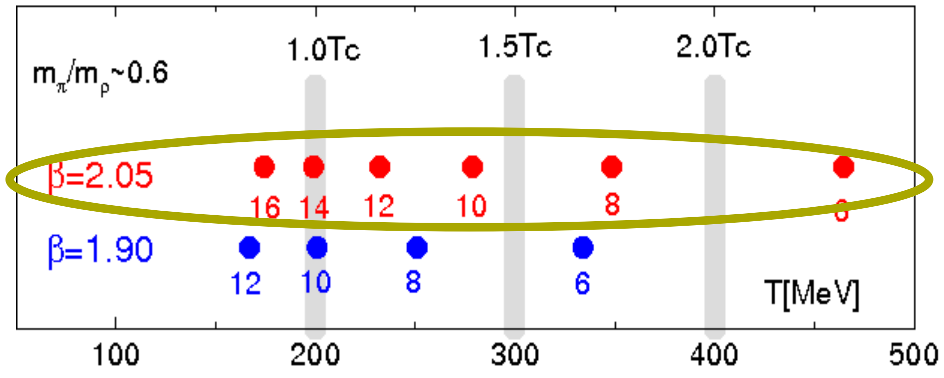
➡ **computation costs much reduced**

# EOS with fixed-scale approach

## Study in Nf=2+1 QCD

Phys. Rev. D 85 (2012) ref.094508, "Equation of state in 2+1 flavor QCD with improved Wilson quarks by the fixed scale approach",
T. Umeda, S. Aoki, S. Ejiri, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno

- T=0 simulation: on $28^3 \times 56$ by CP-PACS/JLQCD *Phys. Rev. D78 (2008) 011502*
  - RG-improved Iwasaki glue + NP-improved Wilson quarks
  - $\beta$=2.05, $\kappa_{ud}$=0.1356, $\kappa_s$=0.1351
  - V~(2 fm)$^3$ , a~0.07 fm,
  - configurations available on the ILDG/JLDG

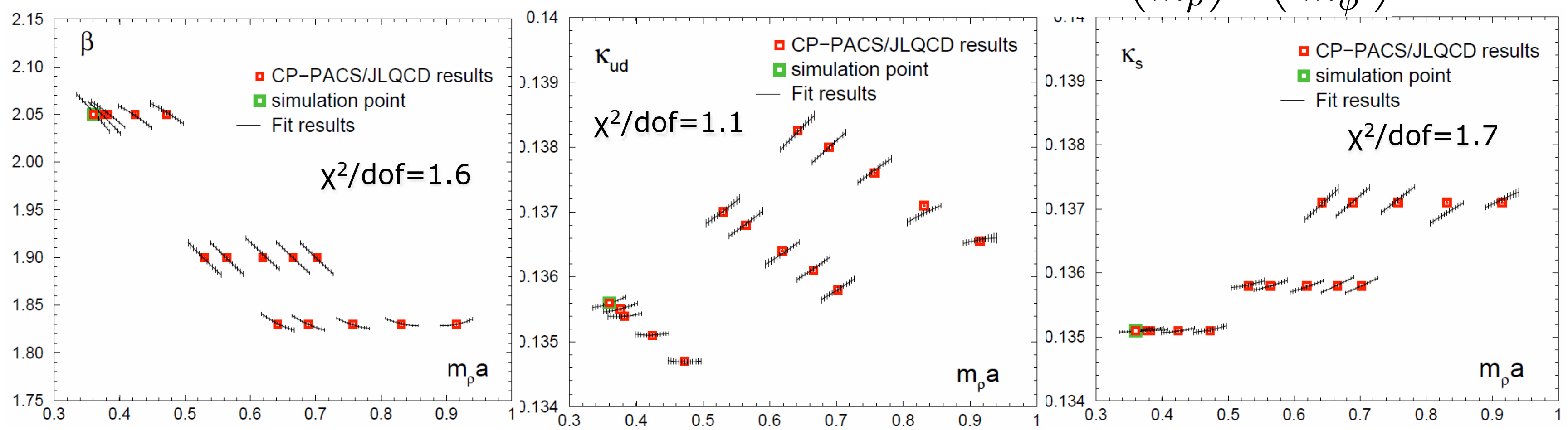



- T>0 simulations: on $32^3 \times N_t$ ($N_t$=4, 6, ..., 14, 16) lattices

RHMC algorithm, same parameters as T=0 simulation

At the lightest point on the finest lattice among 30 simulation points of CP-PACS/JLQCD.

Beta functions by fitting $\beta,\kappa_{ud},\kappa_s$ as functions of $(am_\rho), \left(\frac{m_\pi}{m_\rho}\right), \left(\frac{m_{\eta ss}}{m_\phi}\right)$ [direct fit method]

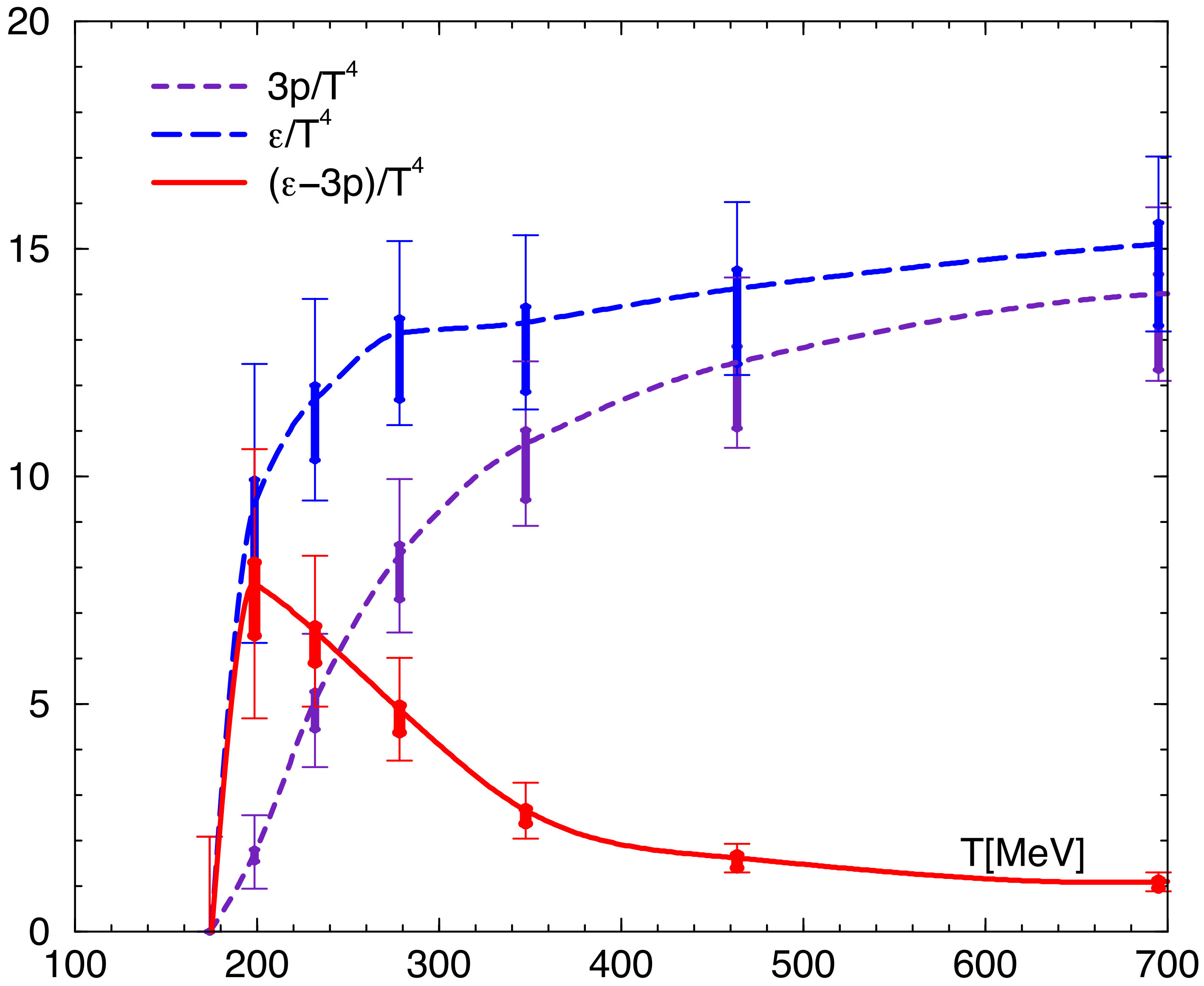



$$\left(a\frac{\partial\beta}{\partial a},\ a\frac{\partial\kappa_{ud}}{\partial a},\ a\frac{\partial\kappa_s}{\partial a}\right)_{\text{simulation point}} = \left(-0.279(24)(^{+40}_{-64}), 0.00123(41)(^{+56}_{-68}), 0.00046(26)(^{+42}_{-44})\right)$$

# EOS with fixed-scale approach

## Study in Nf=2+1 QCD

Phys. Rev. D 85 (2012) ref.094508, "Equation of state in 2+1 flavor QCD with improved Wilson quarks by the fixed scale approach",
T. Umeda, S. Aoki, S. Ejiri, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno




roughly consistent with EOS from highly improved stag. quarks

FIG. 8 (color online). Trace anomaly $(\epsilon - 3p)/T^4$, energy density $\epsilon/T^4$, and pressure $3p/T^4$ in $2+1$ flavor QCD. The thin and thick vertical bars represent statistic and systematic errors, respectively. The curves are drawn by the Akima spline interpolation.

The first Nf=2+1 EOS with Wilson-type quarks.

# Search for the critical point with histogram method

To extend the study to μ≠0, we first have to clarify the phase structure.
To efficiently detect the phase, we developed a method based on the histogram.

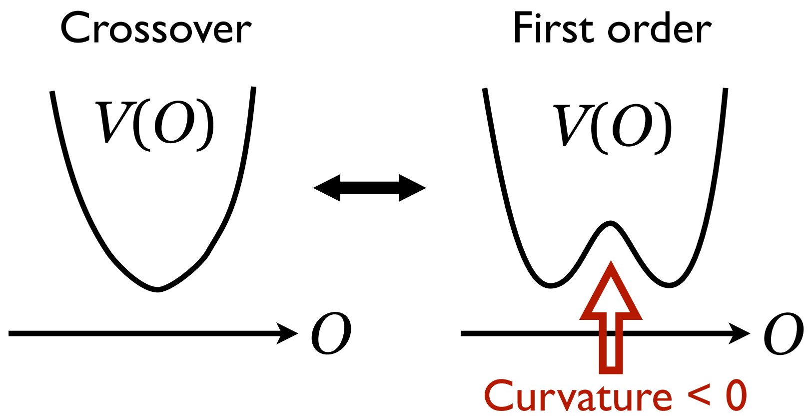
## **Histogram and effective potential** of observables

$$w(\mathcal{O}_1, \mathcal{O}_2, \cdots; \beta, m, \mu) \stackrel{\text{def.}}{=} \int DU \prod_i \delta(\hat{\mathcal{O}}_i[U] - \mathcal{O}_i) \, [\det M(m,\mu)]^{N_F} e^{-S_g(\beta)}$$

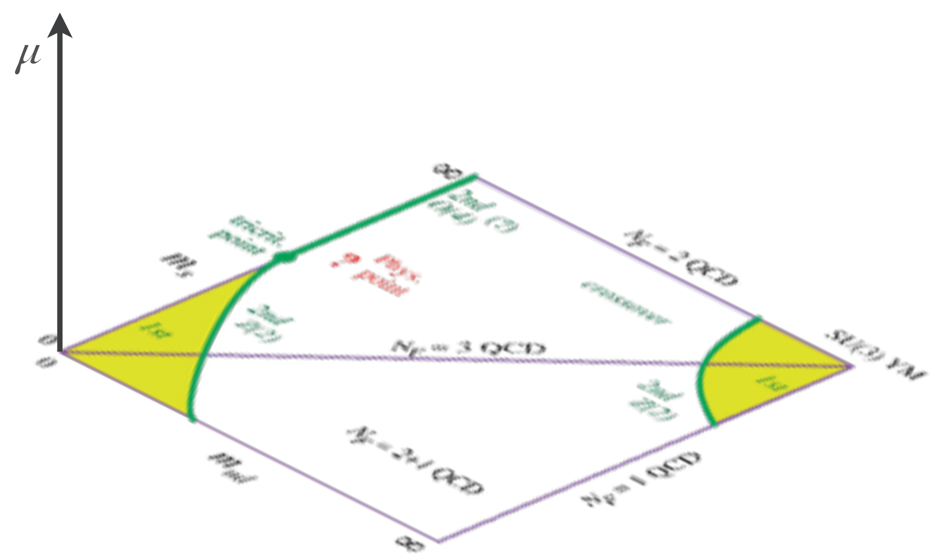
$$Z_{(\beta,m,\mu)} = \int w(\mathcal{O}_1, \cdots; \beta, m, \mu) \prod_i d\mathcal{O}_i \qquad \left\langle f(\hat{\mathcal{O}}_1, \cdots) \right\rangle_{(\beta,m,\mu)} = \frac{1}{Z_{(\beta,m,\mu)}} \int f(\mathcal{O}_1, \cdots) \, w(\mathcal{O}_1, \cdots; \beta, m, \mu) \prod_i d\mathcal{O}_i$$

$$V_{\text{eff}}(\hat{\mathcal{O}}_1, \cdots; \beta, m, \mu) \stackrel{\text{def.}}{=} -\ln w(\mathcal{O}_1, \cdots; \beta, m, \mu)$$

Choosing
we can detect the phase transition through




Critical boundary of
1st order region



# Search for the critical point with histogram method

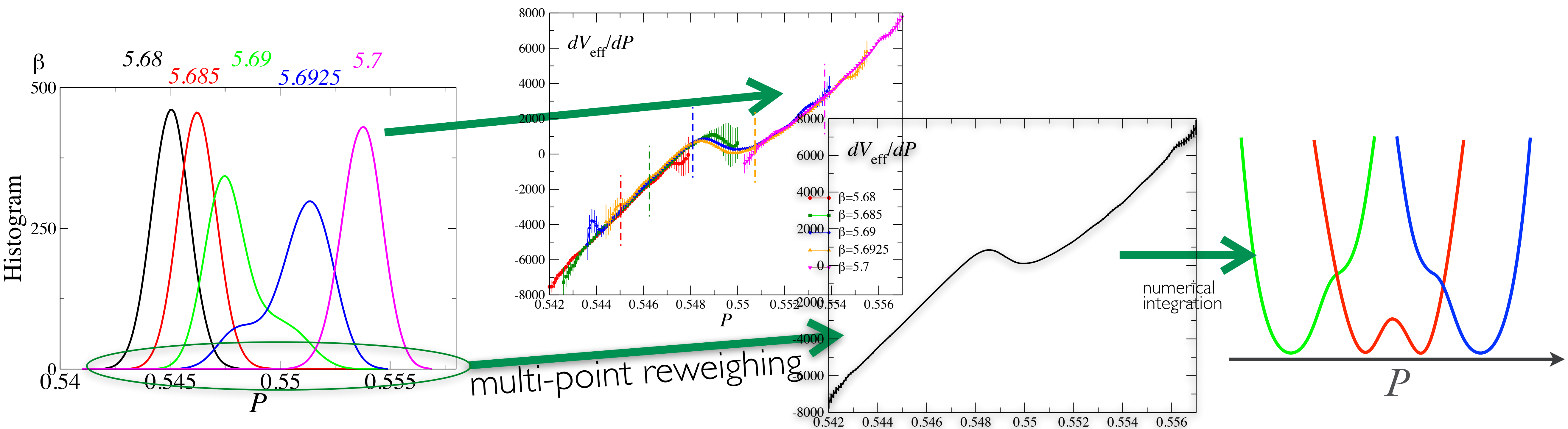
We need reliable $V_{\rm eff}$ in a wide range of $\mathcal{O}$ around the transition pt.

<= **reweighing technique** & **appropriate choice of $\mathcal{O}$**

Pure gauge theory

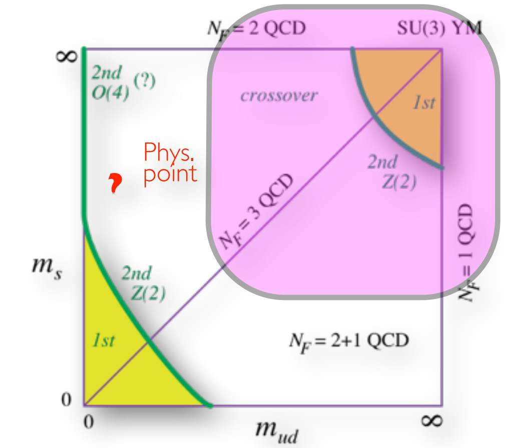
$$w(P,\cdots;\beta,m,\mu) = \int DU\, \delta(\hat{P}[U]-P)\cdots[\det M(m,\mu)]^{N_F} e^{6\beta N_{\rm site} P}$$

$$\begin{cases} V_{\rm eff}(P,\cdots;\beta',m,\mu) = V_{\rm eff}(P,\cdots;\beta,m,\mu) - 6(\beta'-\beta)N_{\rm site}P \\ \dfrac{\partial}{\partial P}V_{\rm eff}(P,\cdots;\beta',m,\mu) = \dfrac{\partial}{\partial P}V_{\rm eff}(P,\cdots;\beta,m,\mu) - \underbrace{6(\beta'-\beta)N_{\rm site}}_{\text{constant, known shift}} \\ \dfrac{\partial^2}{\partial P^2}V_{\rm eff}(P,\cdots;\beta',m,\mu) = \dfrac{\partial^2}{\partial P^2}V_{\rm eff}(P,\cdots;\beta,m,\mu) \end{cases}$$




Phys. Rev. D 84 (2011) ref.054502, "Phase structure of finite temperature QCD in the heavy quark region", H. Saito, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno, T. Umeda

# Search for the critical point with histogram method

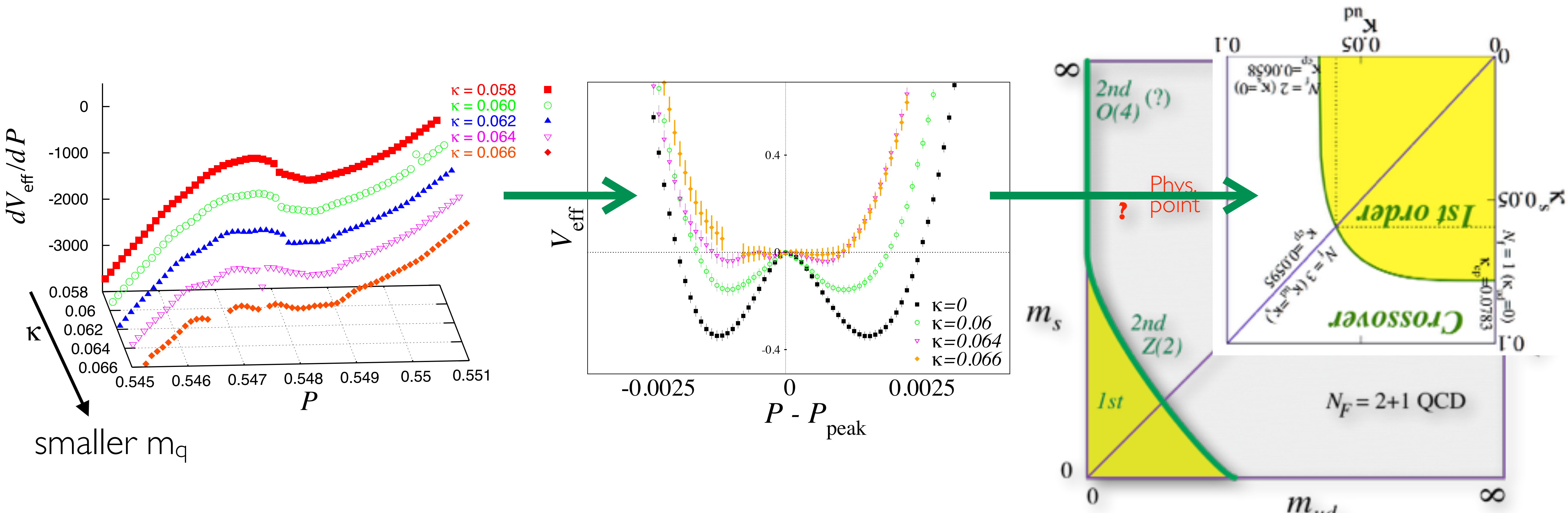



## Heavy quark QCD at μ=0:

Heavy quark limit = SU(3) YM theory with 1st order deconf. trans.

Reweight from the heavy quark limit, using the hopping param. expansion.

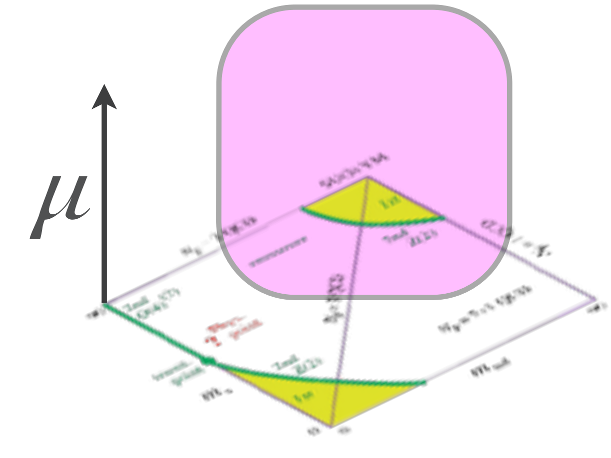
$$\left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{288 N_{\rm site}\kappa^4 \hat{P} + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\,\hat{\Omega}_{\rm R} + i\sinh(\mu/T)\,\hat{\Omega}_{\rm I}\right) + \cdots\right\}\right]$$




Phys. Rev. D 84 (2011) ref.054502, "Phase structure of finite temperature QCD in the heavy quark region", H. Saito, S. Ejiri, S. Aoki, T. Hatsuda, K. Kanaya, Y. Maezawa, H. Ohno, T. Umeda

# Search for the critical point with histogram method

## Heavy quark QCD at μ≠0:




Reweight to μ≠0

$$V_{\rm eff}(\mathcal{O};\beta,\kappa,\mu) = V_{\rm eff}(\mathcal{O};\beta,0,0) - \ln\left\langle \left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F}\right\rangle_{\mathcal{O};\beta,0,0}$$

$$\left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{288 N_{\rm site}\kappa^4\hat{P} + 12\cdot 2^{N_t}N_s^3\kappa^{N_t}\left(\cosh(\mu/T)\,\hat{\Omega}_{\rm R} + i\sinh(\mu/T)\,\hat{\Omega}_{\rm I}\right) + \cdots\right\}\right]$$

Choose

$$V_{\rm eff}(\Omega_{\rm R};\beta,\kappa,\mu) = \boxed{V_{\rm eff}(\Omega_{\rm R};\beta^*,0,0) - 12\cdot 2^{N_t}N_F N_s^3\kappa^{N_t}\cosh(\mu/T)\,\Omega_{\rm R}} - \ln\left\langle e^{i\hat{\theta}}\right\rangle_{\Omega_{\rm R};\beta,0,0}$$

Phase-quenched QCD

$$\hat{\theta} = 12\cdot 2^{N_t}N_F N_s^3\kappa^{N_t}\sinh(\mu/T)\,\hat{\Omega}_{\rm I}$$

Phase-quenched QCD is simple:
Critical line in the phase-quenched QCD is just given by

$$\kappa_{\rm cp}^{\rm p.q.}(\mu) \;=\; \kappa_{\rm cp}(\mu{=}0)\cdot[\cosh(\mu/T)]^{-1/N_t}$$

# Search for the critical point with histogram method

$$V_{\rm eff}(\Omega_{\rm R};\beta,\kappa,\mu) = V_{\rm eff}(\Omega_{\rm R};\beta^*,0,0) - 12\cdot 2^{N_t} N_F N_s^3 \kappa^{N_t}\cosh(\mu/T)\,\Omega_{\rm R} - \ln\left\langle e^{i\hat\theta}\right\rangle_{\Omega_{\rm R};\beta,0,0}$$

To study the effects of the phase term,

**Cumulant expansion**

S. Ejiri, PRD *77*, 014508 ('08); WHOT, PRD 82, 014508 ('10)

$$\langle e^{i\hat\theta}\rangle = \exp\left[i\langle\hat\theta\rangle_c - \frac{1}{2!}\langle\hat\theta^2\rangle_c - \frac{i}{3!}\langle\hat\theta^3\rangle_c + \frac{1}{4!}\langle\hat\theta^4\rangle_c + \cdots\right]$$

$$\langle\hat\theta\rangle_c = \langle\hat\theta\rangle,\quad \langle\hat\theta^2\rangle_c = \langle\hat\theta^2\rangle - \langle\hat\theta\rangle^2,\quad \langle\hat\theta^3\rangle_c = \langle\hat\theta^3\rangle - 3\langle\hat\theta^2\rangle\langle\hat\theta\rangle + 2\langle\hat\theta\rangle^3,\ \cdots$$

❧ Odd terms vanish due to the symmetry under μ

=> $\langle e^{i\hat\theta}\rangle$ positive definite. => **Sign problem resolved if the expansion converges.**

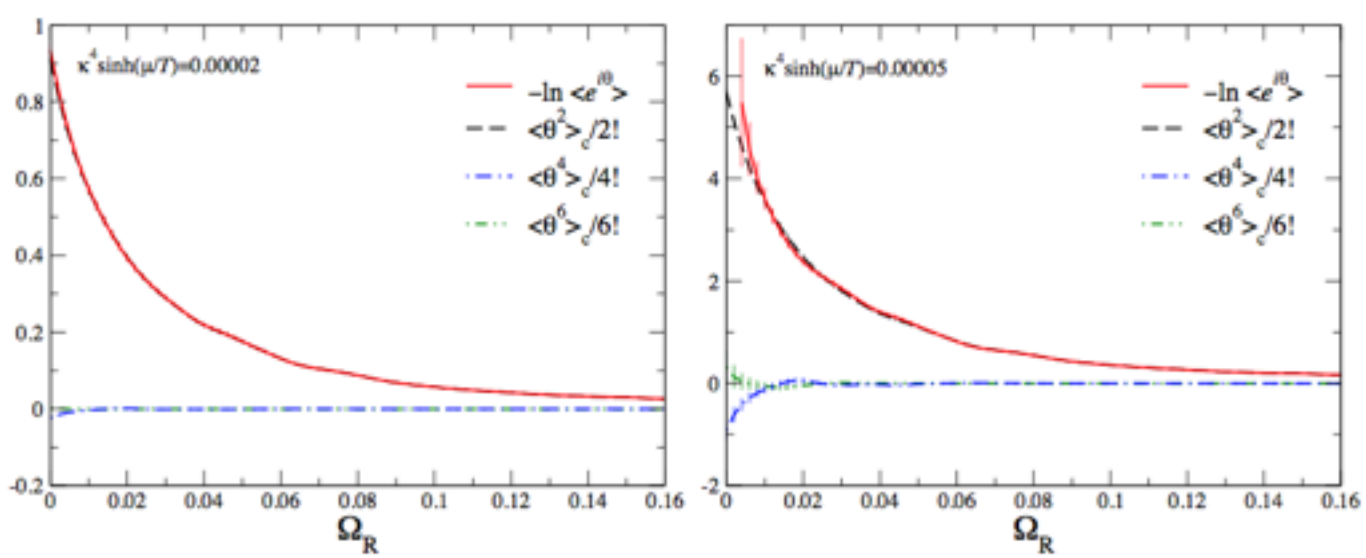



FIG. 9: Exponent of the average phase factor, $-\ln\langle e^{i\hat\theta}\rangle$, compared with the contributions from the 2nd, 4th and 6th order cumulants. The expectation values are calculated at $\beta^* = 5.69$ and $\kappa^4\sinh(\mu/T) \approx 0.00002$ (left) or 0.00005 (right) in $N_f = 2$ QCD with fixed $\Omega_{\rm R}$. In both cases, $-\ln\langle e^{i\hat\theta}\rangle$ is almost indistinguishable with $\langle\hat\theta^2\rangle_c/2!$.

We find,

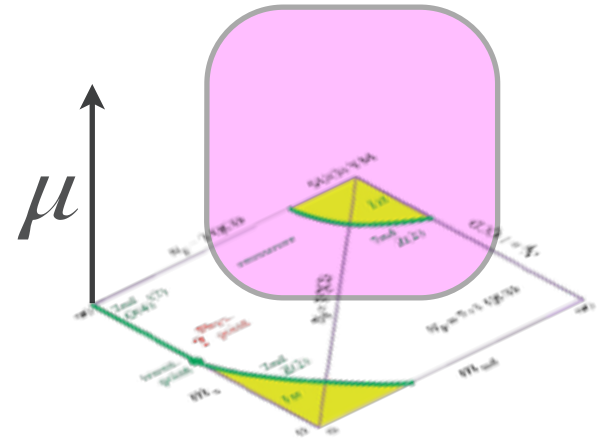
around the critical line,

**Gaussian term dominates in** $\langle e^{i\hat\theta}\rangle$

i.e. most convergent!

# Search for the critical point with histogram method

## Heavy quark QCD at μ≠0:



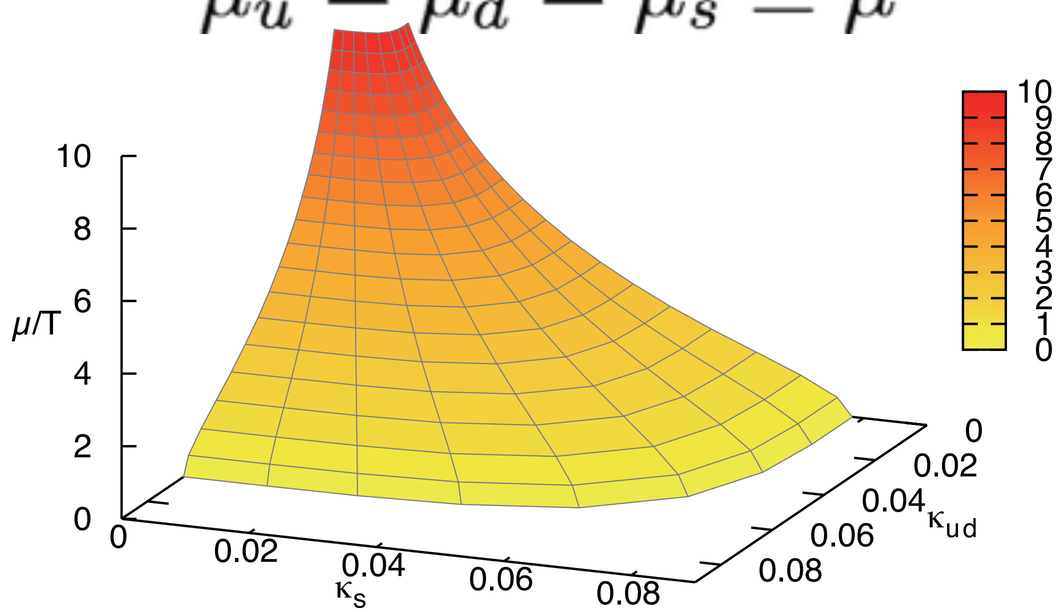

We can now reliably evaluate $\langle e^{i\hat{\theta}} \rangle$ by the cumulant expansion around the crit. line.

=> The effects turned out to be quite small on the crit. line at all μ.

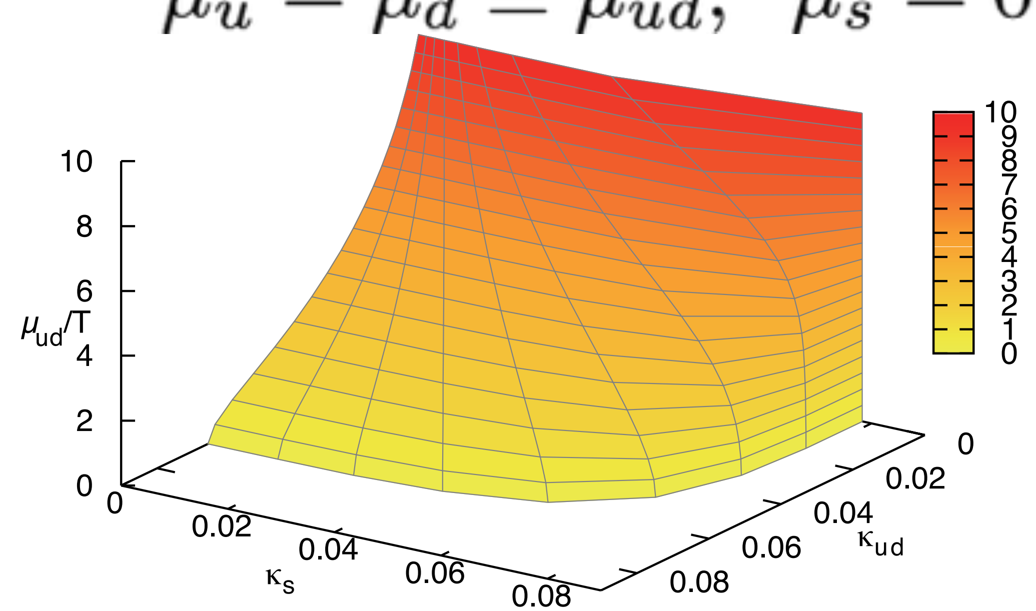
=> **The critical line well estimated by the phase-quenched approx.**

### ❧ Critical surface in heavy quark QCD

$$\mu_u = \mu_d = \mu_s \equiv \mu$$




$$\mu_u = \mu_d \equiv \mu_{ud}, \quad \mu_s = 0$$




Phys. Rev. D (2014) in press, "Histograms in heavy-quark QCD at finite temperature and density", H. Saito, S. Ejiri, S. Aoki, K. Kanaya, Y. Nakagawa, H. Ohno, K. Okuno, T. Umeda

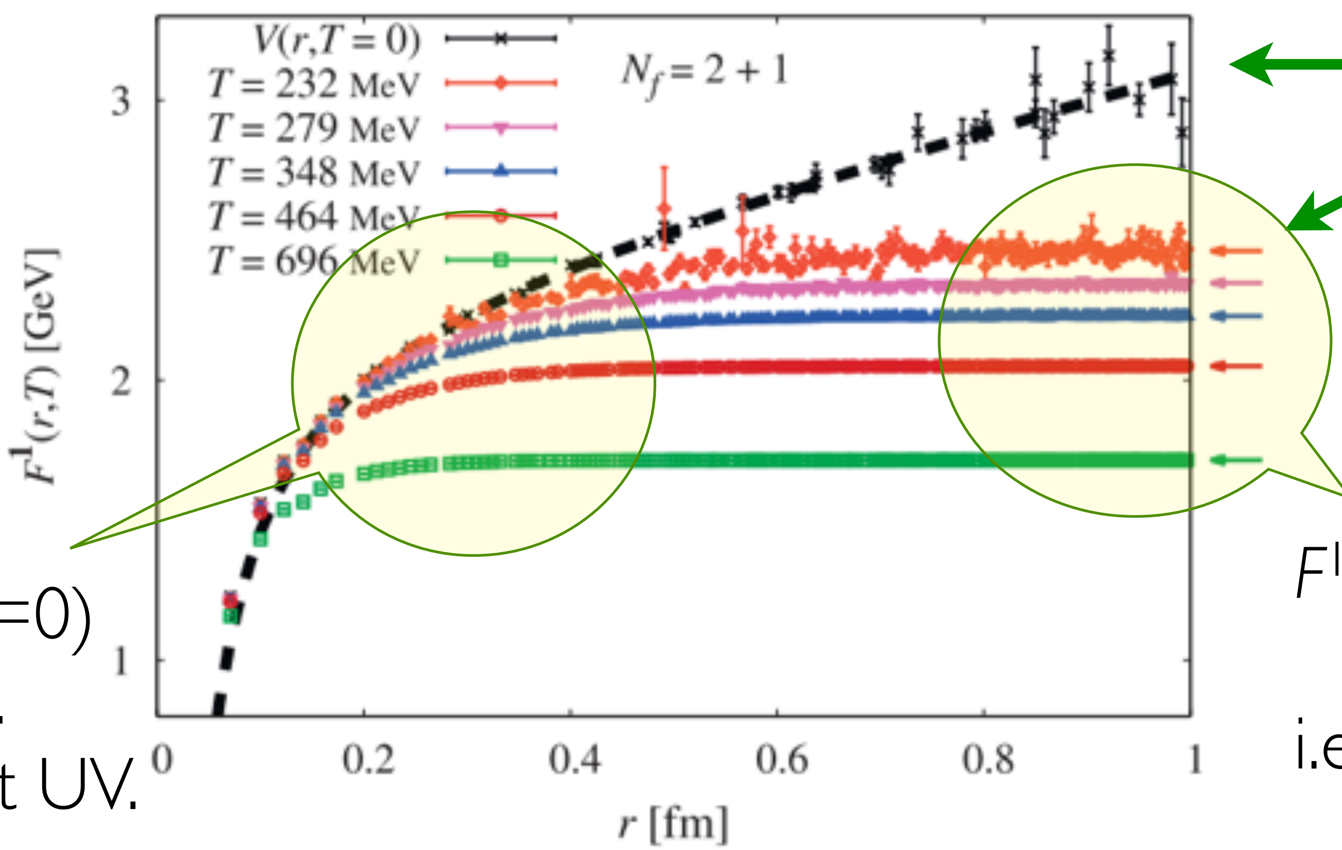
# $Q\text{-}\overset{(-)}{Q}$ interaction and screening masses at finite T and μ

Phys. Rev. D 81 (2010) ref.091501(R), "Electric and Magnetic Screening Masses at Finite Temperature from Generalized Polyakov-Line Correlations in Two-flavor Lattice QCD", Y. Maezawa, S. Aoki, S. Ejiri, T. Hatsuda, N. Ishii, K. Kanaya, N. Ukita and T. Umeda

Phys. Rev. D 82 (2010) ref.014508, "Equation of State and Heavy-Quark Free Energy at Finite Temperature and Density in Two Flavor Lattice QCD with Wilson Quark Action ", S. Ejiri, Y. Maezawa, N. Ukita, S. Aoki, T. Hatsuda, N. Ishii, K. Kanaya, T. Umeda

Prog. Theor. Phys. 128 (2012), "Application of fixed scale approach to static quark free energies in quenched and 2 + 1 flavor lattice QCD with improved Wilson quark action", Y. Maezawa, T. Umeda, S. Aoki, S. Ejiri, T. Hatsuda, K. Kanaya and H. Ohno

- Heavy quark free energy at $T>T_c$ $\quad F^1(r,T) = -T\ln\langle \mathrm{Tr}\,\Omega^\dagger(\mathbf{x})\Omega(\mathbf{y})\rangle$ in the Coulomb gauge




$T$=0 data by CP-PACS+JLQCD

$2F_Q = -2T\ln\langle \mathrm{Tr}\,\Omega\rangle$

2 x single quark free energy

$F^1$ —> $V(r) = F^1(T=0)$
at short distances.
i.e. T-insensitivity at UV.

$F^1$ —> $2F_Q$
at long distances.
i.e. deconfinement at IR

- **No vertical adjustment needed** in the fixed scale approach.

(cf.) In the fixed-Nt approach, $F^1$ -->$V(r)$ is used as an input to adjust the constant term of $F^1$.

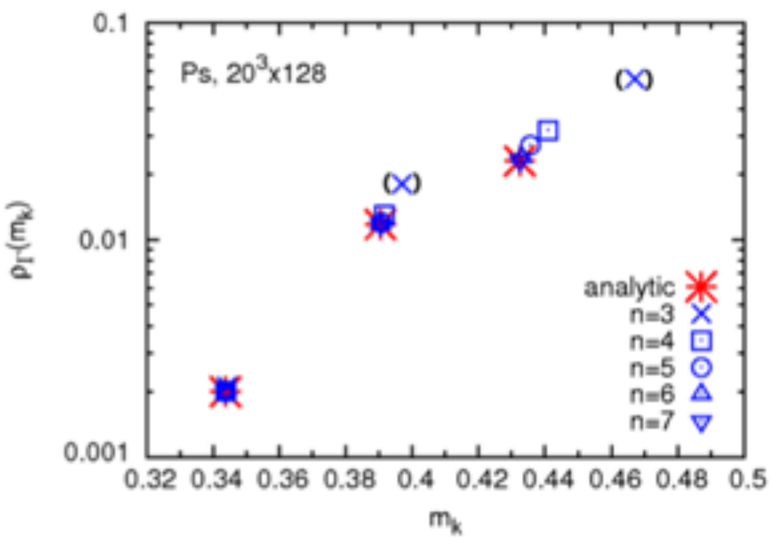
# Charmonium dissociation with variational method

Phys. Rev. D 84 (2011) ref.094504, "Charmonium spectral functions with the variational method in zero and finite temperature lattice QCD", H. Ohno, S. Aoki, S. Ejiri, K. Kanaya, Y. Maezawa, H. Saito and T. Umeda

Computation of the spectral function (SpF) on the lattice is an ill-posed problem. Conventional estimation using MEM is always heuristic.

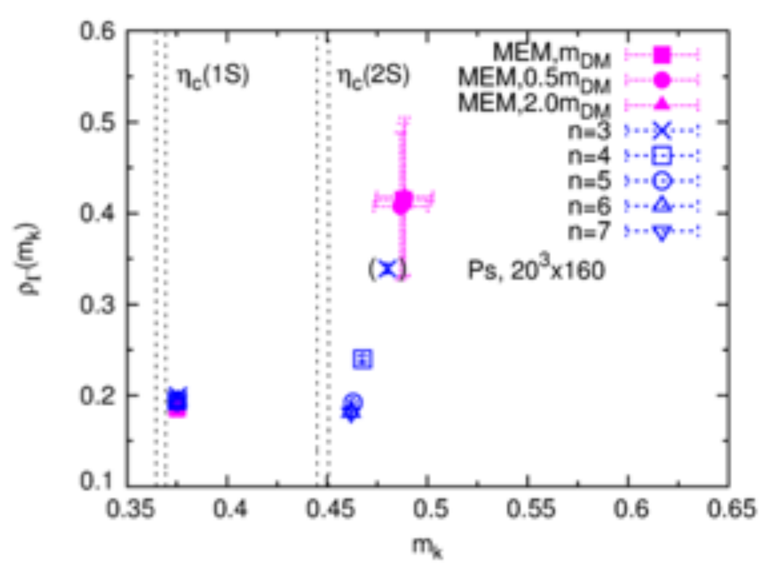
We define an effective SpF in terms of the correlation matrix between a series of smeared operators. This eff. SpF tends to the correct SpF on the lattice at large distances, and the systematic error due to finite distance can be reduced by increasing the number of operators and by a judicious choice of the set of operators.

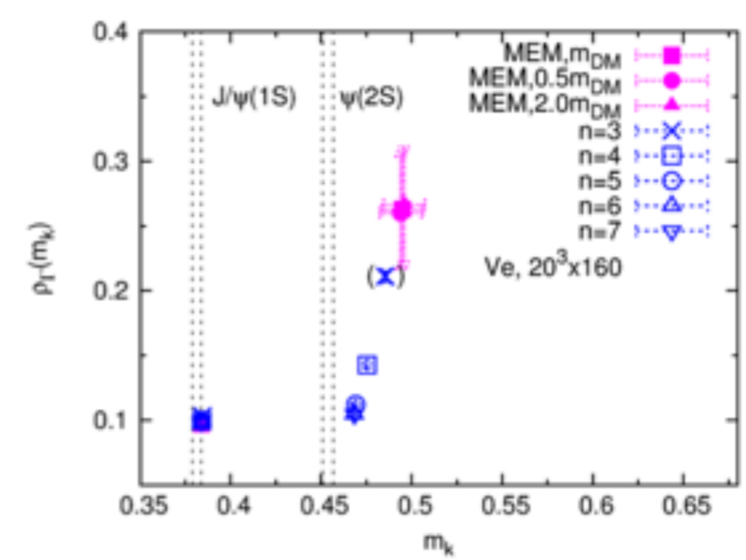
## A test with free quarks




Analytic results for low-lying poles and their width well reproduced by increasing # of sources.

## With dynamical quarks




Ground state consistent with experiment as well as the MEM result. 1st exited state deviates from the MEM estimation and gets closer to the experiment with increasing # of sources.

# Our main results since 2008

- EOS at $\mu \neq 0$ with dynamical Wilson quarks

  Cumulant expansion useful at small $\mu$.
  The first EOS at $\mu \neq 0$ with Wilson-type quarks.

- EOS with fixed-scale approach

  Fixed-scale approach with T-integration method works.
  The first $N_f=2+1$ EOS with Wilson-type quarks.

- Search for the critical point with histogram method

  Histogram + reweighting powerful for the phase structure.
  Test in heavy quark QCD.

- Q-$\overset{(-)}{Q}$ interaction and screening masses at finite T and $\mu$

  Q-$\bar{Q}$ int. weaker at $\mu \neq 0$, Q-Q int. stronger at $\mu \neq 0$.
  Fixed-scale approach allows us a direct test of T-dependence.

- Charmonium dissociation with variational method

  A solid method to compute spectral function on the lattice.
  Works well for low-lying states (pole position and width).

# Plans for the next stage

## Nf=2+1 EOS at the physical point

Extend the EOS study at μ=0 towards the physical point on a fine lattice.

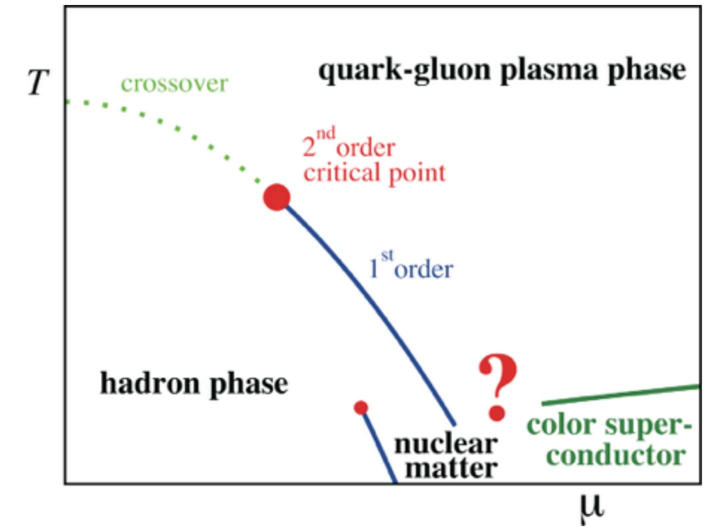
* Fixed-scale approach making use of T=0 PACS-CS or HAL configurations.
* Need beta functions <= reweighing technique?

## Critical point at μ≠0

Extend the phase structure study to light quark QCD.

* Histogram method with dynamical quarks.
* Can detect the critical point?

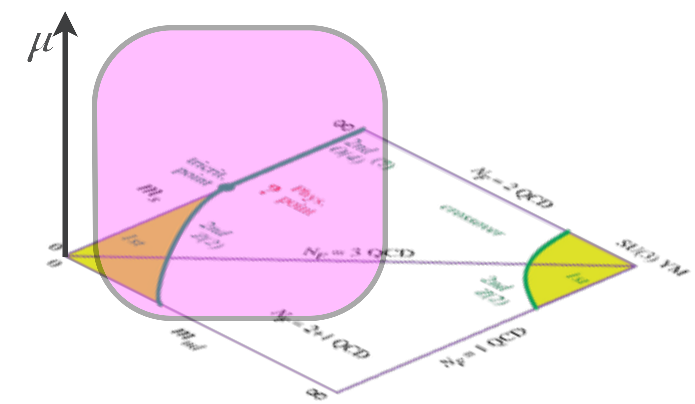
=> See below.




## Scaling properties at μ=0 and ≠0

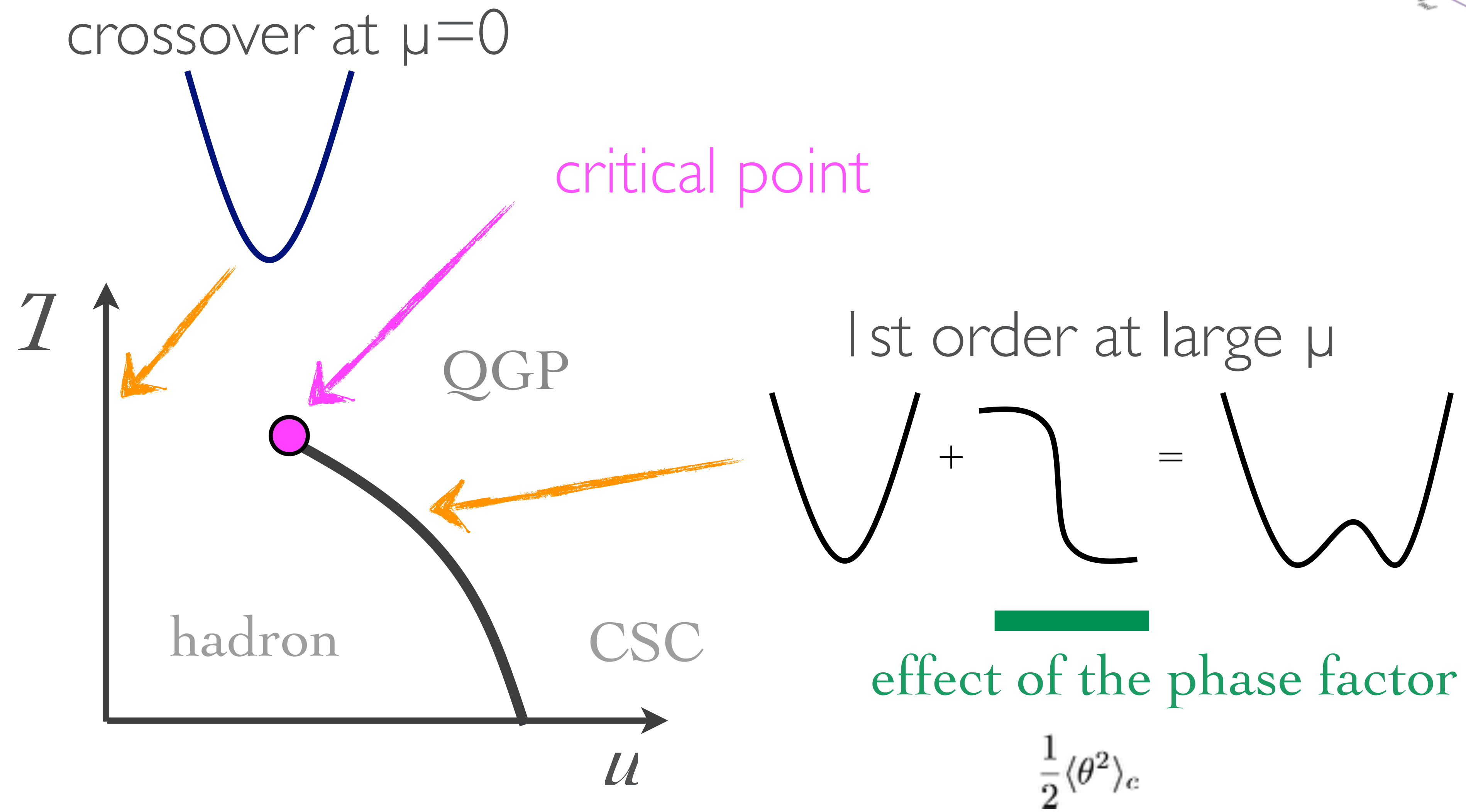
* Tricritical point, curvature of the pseudo critical line, …
* O(4)? or something else?

# Search for the critical point with histogram method



## QCD with light quarks:

We expect for 2-flavor QCD with




We try to find the crit. pt. through negative curvature in V

# Search for the critical point with histogram method

## QCD with light quarks:

Our strategy:

* Phase-quenched simulation with $|\det M|^{N_f}\, e^{-S_g}$
* Cumulant expansion for the phase $e^{i\theta}$ with $\theta(\mu) = N_{\rm f} \int_0^{\mu/T} \mathfrak{Im}\left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)}\right]_{\bar\mu} d\left(\frac{\bar\mu}{T}\right)$
* To cover a wide range of $V_{eff}$ by another reweighting, we choose

$P = -S_g/6\beta N_{site}$ ≈ glue energy

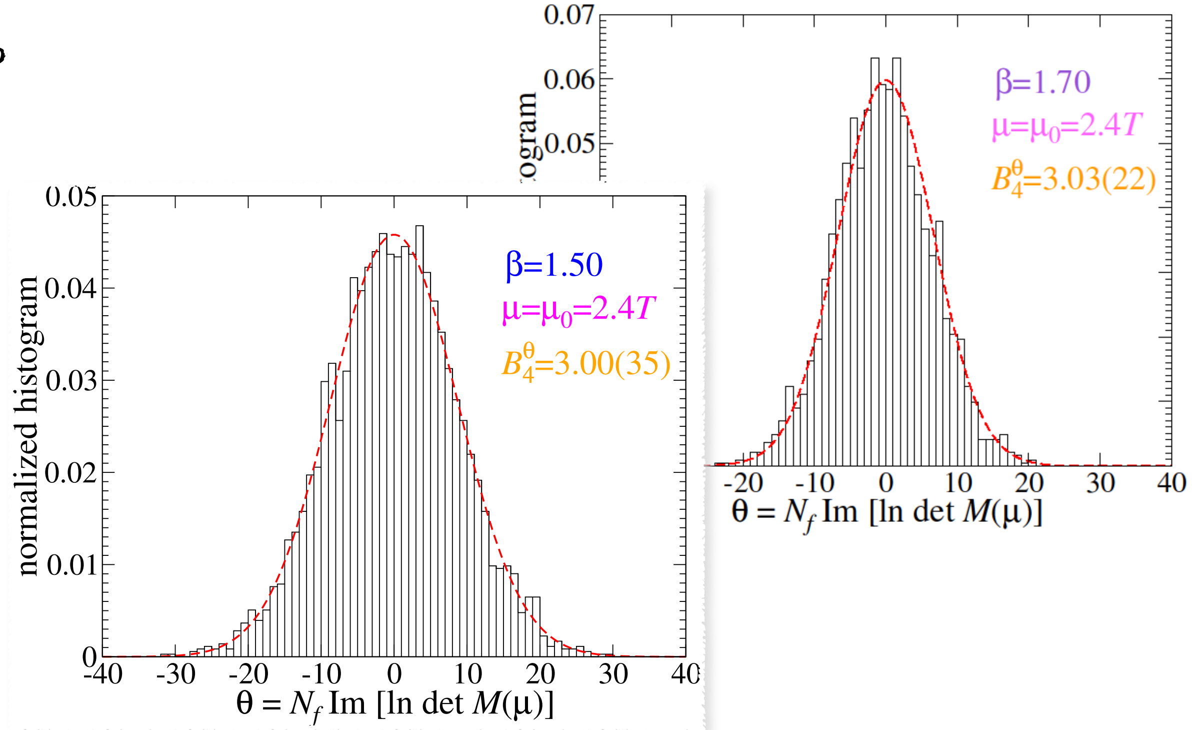
$F = N_f \ln|\det M(\mu)/\det M(0)|$ ≈ quark energy

as O's for $V_{eff}$

$$Z(\beta,\mu) = \int \mathscr{D}U\, e^{i\theta(\mu)}\, |\det M(\mu)|^{N_{\rm f}}\, e^{6\beta N_{\rm site} P}$$

We again find Gaussian dominance at small μ/T.

$$\left\langle e^{i\theta}\right\rangle_{P,F} \approx \exp\left[-\frac{1}{2}\left\langle \theta^2\right\rangle\right]$$

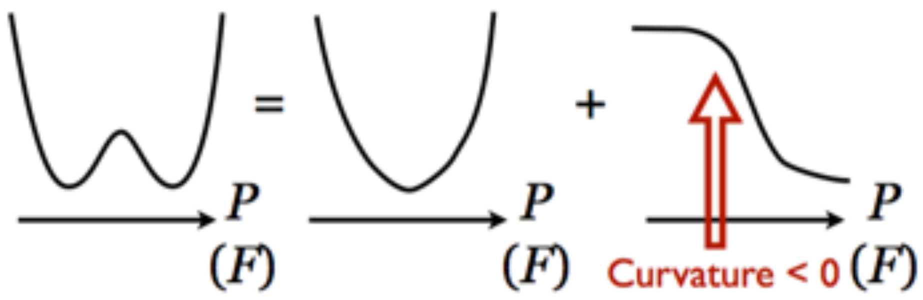


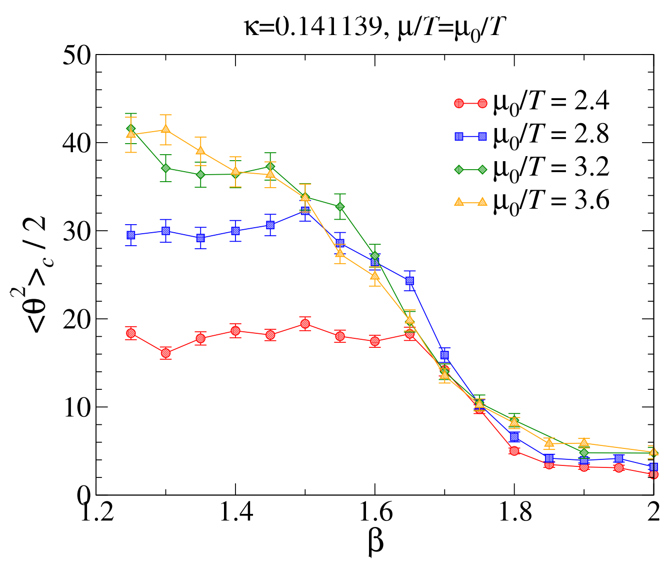

Can we reach the crit. point?

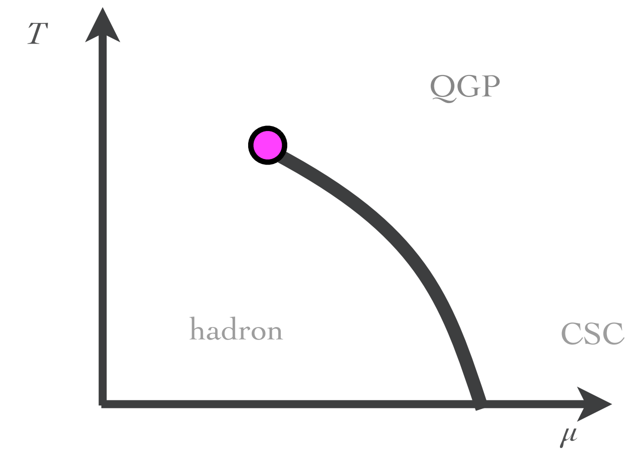
# Search for the critical point with histogram method

QCD with light quarks:

$$V(P,F;\beta,\mu) = -\ln w(P,F;\beta,\mu_0) + \frac{1}{2}\langle\theta^2\rangle_c(P,F;\beta,\mu,\mu_0)$$

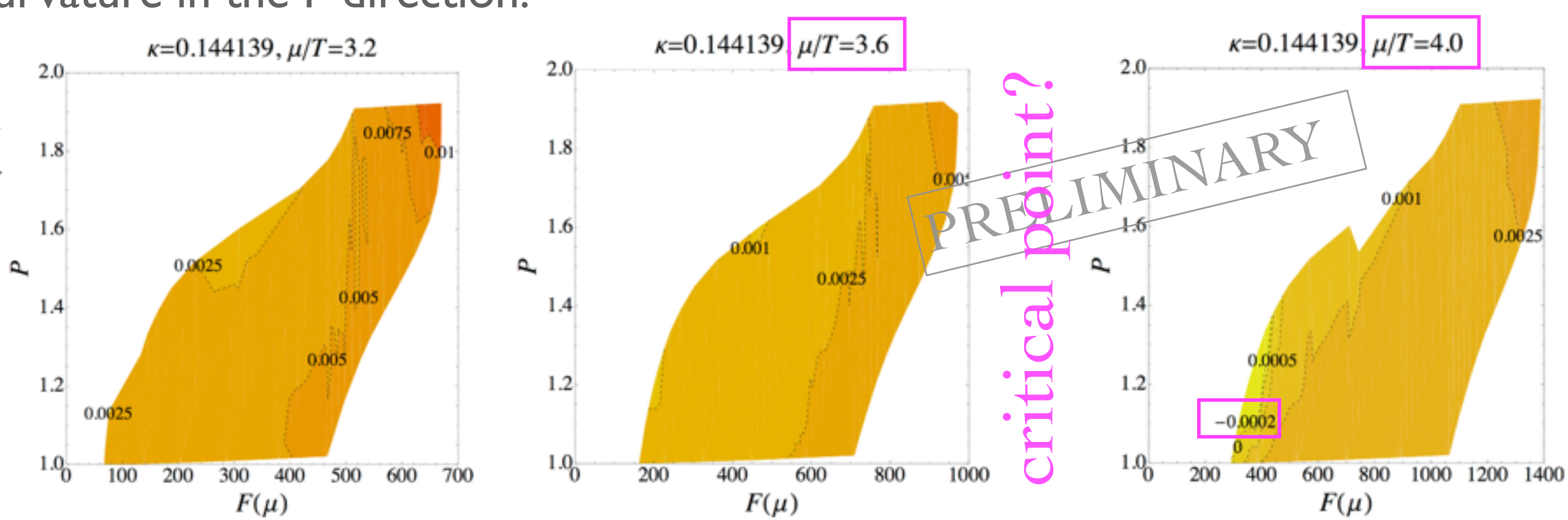



- Curvature in the F-direction:

$$\frac{\partial^2 V}{\partial F^2}$$



thank you