

**Schrödinger functional  
formalism  
with  
Ginsparg-Wilson fermion**

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# §1 Introduction

- Schrödinger functional (Lüscher et al.,,,)

$$Z = \langle C'; T | C; 0 \rangle = \int \mathcal{D}\Phi e^{-S[\Phi]}$$

- renormalization scheme
- finite box  $L^3 \times T \sim L^4$
- Dirichlet boundary condition

$$A_k|_{x_0=0} = C_k, \quad A_k|_{x_0=T} = C'_k,$$
$$P_+\psi|_{x_0=0} = \rho, \quad P_-\psi|_{x_0=T} = \rho', \quad P_{\pm} = \frac{1 \pm \gamma_0}{2}$$

⇓

**Renormalizable** (Lüscher et al, Sint)

- finite mass gap  $\sim 1/T$

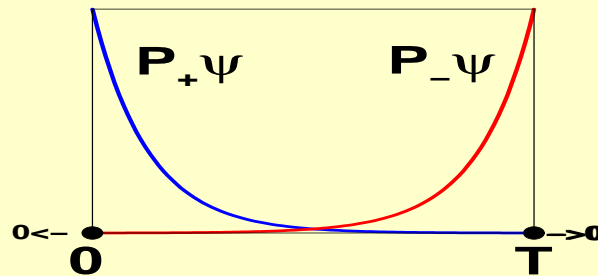
- A potential problem with Dirichlet BC  $\Rightarrow$  **zero mode**

- zero eigenvalue equation

$$\mathcal{L} = \bar{\psi} (\gamma_\mu \partial_\mu + m) \psi \quad \Rightarrow \quad (\gamma_0 \partial_0 + m) \psi = 0$$

- under boundary condition

$$P_- \psi|_{x_0=0} = 0, \quad P_+ \psi|_{x_0=T} = 0$$



$$\psi = P_+ e^{-mx_0} + P_- e^{-m(T-x_0)}$$

SF BC :  $P_+ \psi|_{x_0=0} = 0, P_- \psi|_{x_0=T} = 0 \Rightarrow$  forbids zero mode

- Wilson fermion (Sint)

- DBC  $\Leftrightarrow$  Wilson parameter  $r = \pm 1$

$$\begin{aligned}
 D_W &= \gamma_\mu \nabla_\mu - \frac{r}{2} \Delta + M \\
 &= -\frac{1 - \gamma_0}{2} \delta_{m_0, n_0+1} - \frac{1 + \gamma_0}{2} \delta_{m_0, n_0-1} + D_W^{(3)} + (M + 1) \\
 &= \begin{pmatrix} D_W^{(3)} + M + 1 & & -P_- \\ -P_+ & D_W^{(3)} + M + 1 & \\ & -P_+ & D_W^{(3)} + M + 1 \end{pmatrix}
 \end{aligned}$$



$$\text{SF BC : } P_+ \psi|_{x_0=0} = 0, P_- \psi|_{x_0=T} = 0$$

- $M \geq 0$  to forbid zero mode solution
- Problem may become fatal in overlap Dirac operator

- Overlap Dirac operator (Neuberger)

$$aD = 1 + D_W \frac{1}{\sqrt{D_W^\dagger D_W}}$$

$$D_W = -\frac{1 - \gamma_\mu}{2} \delta_{m, n+\mu} - \frac{1 + \gamma_\mu}{2} \delta_{m, n-\mu} + (-M + 4)$$

Signature of  $r$ ,  $M$  fixed to eliminate doubler  
(for  $r = 1$  ;  $-1 < 1 - M < 1$ )

⇒ zero mode solution is allowed in  $D_W$

$$\text{SF BC : } P_+ \psi|_{x_0=0} = 0, P_- \psi|_{x_0=T} = 0$$

$$\psi = P_- (1 - M)^{x_0} + P_+ (1 - M)^{(T-x_0)}$$

⇒ non-locality ⇒ Dirichlet BC in  $D_W$  does not work!

- Domain-wall fermion (Kaplan, Shamir)

$$D = \gamma_M \nabla_M - \frac{1}{2} \Delta^2 - M$$

Dirichlet BC in 5-th direction

$$P_R \psi|_{x_5=0} = 0, \quad P_L \psi|_{x_5=N_5} = 0, \quad P_{R/L} = \frac{1 \pm \gamma_5}{2}$$

⇒ zero mode solution in 5-th direction

$$\psi = P_L (1 - M)^{x_5} + P_R (1 - M)^{(N_5 - x_5)}$$

$M \sim$  smallest eigenvalue of  $\gamma_5 D_W^{(4)}$

SF Dirichlet BC in temporal direction

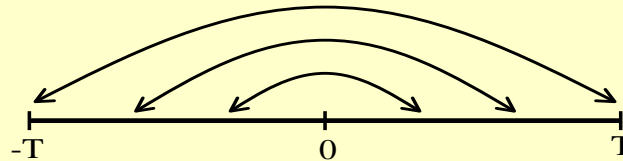
$$P_+ \psi|_{x_0=0} = 0, \quad P_- \psi|_{x_0=T} = 0, \quad P_{\pm} = \frac{1 \pm \gamma_0}{2}$$

⇒ zero mode solution in  $D_W^{(4)}$

## §2 Orbifolding

- A criterion to introduce Dirichlet BC in  $D_{OV}$ ,  $D_{dwf}$ 
  - chiral symmetry breaking by BC :  $(1 + \gamma_0)\psi|_{x_0=0}$   
 $\Rightarrow$  generate mass gap
- field theory with BC  $\Leftrightarrow$  orbifolded field theory

orbifolding:  $S^1/Z_2$   
 $x_0 \leftrightarrow -x_0$



- Identification of fields by symmetry (projection)
  - $\Rightarrow$  to break chiral symmetry
  - $\Rightarrow$  to produce SF Dirichlet BC at fixed points

- **Symmetry for field orbifolding**

- **time reversal symmetry**

$$\psi \rightarrow i\gamma_0\gamma_5 R\psi, \quad \bar{\psi} \rightarrow \bar{\psi}i\gamma_0\gamma_5 R, \quad R\psi(x_0) = \psi(-x_0)$$

- **chiral symmetry (massless theory)**

$$\psi \rightarrow i\gamma_5\psi, \quad \bar{\psi} \rightarrow \bar{\psi}i\gamma_5$$

- **anti-periodicity**

$$\psi(x_0 + 2T) = -\psi(x_0), \quad R\psi(0) = \psi(0), \quad R\psi(T) = -\psi(T)$$

- **Orbifolding**  $\psi(x) = -\gamma_0 R\psi(x), \quad \bar{\psi}(x) = \bar{\psi}(x)\gamma_0 R$

- **SF Dirichlet BC at fixed points**

$$(1 + \gamma_0)\psi(0) = 0, \quad (1 - \gamma_0)\psi(T) = 0$$

$$\bar{\psi}(0)(1 - \gamma_0) = 0, \quad \bar{\psi}(T)(1 + \gamma_0) = 0$$

- **Orbifolded action**

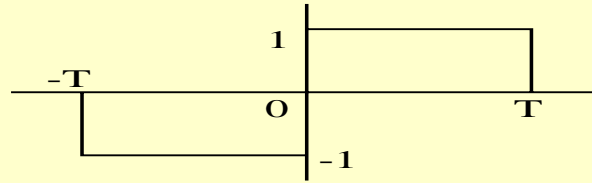
$$S = \int \bar{\psi} D_{\text{SF}} \psi, \quad D_{\text{SF}} = \frac{1 + \Gamma}{2} \not{D} \frac{1 - \Gamma}{2}, \quad \Gamma = \gamma_0 R$$

- Comments

- Gauge fields (external fields)  $\Leftarrow$  SF YM is well defined orbifolding  $A_k(x_0) = A_k(-x_0), \quad A_0(x_0) = -A_0(-x_0)$   
SF DBC  $A_k(0) = C_k, \quad A_k(T) = C'_k, \quad (\partial_0 A_0(0) = 0)$

- Mass term  $\Rightarrow$  should be consistent with orbifolding  $\{M(x), \Gamma\} = 0$

$$M(x) = m\eta(x)$$



- SF Dirac operator

$$D_{\text{SF}} = \frac{1 + \Gamma}{2} (\not{D} + m\eta(x)) \frac{1 - \Gamma}{2}$$

$$D_{\text{SF}} : \mathcal{H}_+ \rightarrow \mathcal{H}_-, \quad D_{\text{SF}}^\dagger : \mathcal{H}_- \rightarrow \mathcal{H}_+$$

$$\mathcal{H}_\pm = \{\psi | (1 \pm \Gamma)\psi = 0\}$$

- Eigenvalue problem (free theory) (Sint)

$$D_{\text{SF}}^\dagger D_{\text{SF}} = \frac{1 - \Gamma}{2} \left( -\partial^2 + m^2 - 2m\gamma_0 (\delta(x_0) - \delta(x_0 - T)) \right) \frac{1 - \Gamma}{2}$$

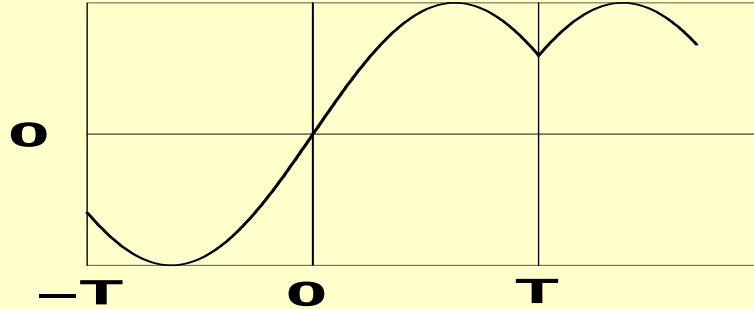
$$D_{\text{SF}}^\dagger D_{\text{SF}} \psi = \lambda^2 \psi, \quad (1 + \Gamma)\psi = 0, \quad \psi(x_0 + T) = -\psi(x_0 - T)$$

- Define  $\psi_\pm \equiv \frac{1 \pm \gamma_0}{2} \psi$

$$\psi_+(x_0) = -\psi_+(-x_0), \quad \psi_+(T + x_0) = \psi_+(T - x_0)$$

$$\psi_+ = A(\sin p_0 x_0) e^{i\vec{p}\vec{x}}$$

$$\lambda^2 = p_0^2 + \vec{p}^2 + m^2$$



- continuity at  $x_0 = 0, T$
- matching of  $\partial_0 \psi$  at  $x_0 = 0, T$

$$-\partial_0 \psi_+ \Big|_{T-\epsilon}^{T+\epsilon} = -2m \psi_+ \Big|_{T-\epsilon}^{T+\epsilon} \quad \Rightarrow \quad \tan p_0 T = -\frac{p_0}{m}$$

$$p_0 = \frac{2n+1}{2T} \pi \quad \text{for } m = 0$$

- Propagator

$$G_{\text{SF}} = 2 \frac{1 - \Gamma}{2} \frac{1}{\not{D} + m\eta(x)} \frac{1 + \Gamma}{2}$$

$$D_{\text{SF}} G_{\text{SF}} = (1 + \Gamma) \delta(x_0 - y_0) = \delta(x_0 - y_0) \quad 0 < x_0, y_0 < T$$

$$(1 + \gamma_0) G_{\text{SF}}|_{x_0=0} = 0, \quad (1 - \gamma_0) G_{\text{SF}}|_{x_0=T} = 0$$

- Massless free propagator

$$G_{\text{SF}} = 2 \not{D}^\dagger \frac{1 + \Gamma}{2} \frac{1}{\not{D} \not{D}^\dagger} \frac{1 + \Gamma}{2} = \not{D}^\dagger (P_+ G_L + P_- G_R)$$

$$G_R(x_0, y_0) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} \frac{1}{p^2} \left( e^{ip_0(x_0 - y_0)} - e^{ip_0(x_0 + y_0)} \right)$$

$$G_L(x_0, y_0) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} \frac{1}{p^2} \left( e^{ip_0(x_0 - y_0)} + e^{ip_0(x_0 + y_0)} \right)$$

$$p_0 = \frac{2n + 1}{2T} \pi$$

⇒ Coincides with that given by Lüscher-Weisz

### §3 Overlap Dirac operator

$$aD = M \left( 1 + D_W \frac{1}{\sqrt{D_W^\dagger D_W}} \right) \quad (\text{Neuberger})$$

- time reversal symmetry

$$\psi \rightarrow i\gamma_0\gamma_5 R\psi, \quad \bar{\psi} \rightarrow \bar{\psi}i\gamma_0\gamma_5 R$$

$$U_k(x_0) = U_k(-x_0), \quad U_0(x_0) = U_0^\dagger(-x_0 - 1) \quad (\text{reflection})$$

$$U_k(0) = W_k, \quad U_k(T) = W_k' \quad (\text{SF Dirichlet BC})$$

- chiral symmetry (Lüscher)

$$\psi \rightarrow i\hat{\gamma}_5\psi, \quad \bar{\psi} \rightarrow \bar{\psi}i\gamma_5, \quad \hat{\gamma}_5 = \gamma_5(1 - aD)$$

- anti-periodicity  $\psi(x_0 + 2T) = -\psi(x_0)$

- Orbifolding projection

$$\psi(x) = -\hat{\Gamma}\psi(x), \quad \bar{\psi}(x) = \bar{\psi}(x)\Gamma, \quad \Gamma = \gamma_0 R, \quad \hat{\Gamma} = \Gamma(1 - aD)$$

- Projection property

$$[i\gamma_5\Gamma, D] = 0, \quad \gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

$$\Rightarrow \Gamma D + D\Gamma = aD\Gamma D, \quad \Gamma^2 = \hat{\Gamma}^2 = 1$$

- Orbifolding

$$(1 + \hat{\Gamma})\psi = 0, \quad \bar{\psi}(1 - \Gamma) = 0 \quad \Rightarrow \quad \text{SF BC in } a \rightarrow 0$$

- Physical quark  $q = (1 - \frac{a}{2}D)\psi, \quad \bar{q} = \bar{\psi}$

$$(1 + \Gamma)q = 0, \quad \bar{q}(1 - \Gamma) = 0$$

- Orbifolded action

$$S = \sum \bar{\psi} D_{\text{SF}} \psi, \quad D_{\text{SF}} = \frac{1 + \Gamma}{2} \left( D + m\eta(x_0) \left( 1 - \frac{a}{2}D \right) \right) \frac{1 - \hat{\Gamma}}{2}$$

- No index

$$\text{tr}\Gamma = \text{tr}\hat{\Gamma} = 0 \quad \Leftarrow \quad [i\gamma_5\Gamma, D] = 0$$

$$\frac{1 \pm \Gamma}{2} \Leftrightarrow \frac{1 \pm \hat{\Gamma}}{2} \quad : \quad \hat{\Gamma} = u^\dagger \Gamma u \quad (u : \text{local, unitary})$$

- $D_{\text{SF}}$  is local

- Eigenvalue problem

$$D_{\text{SF}} = \frac{1 + \Gamma}{2}(D + m\eta u) \frac{1 - \hat{\Gamma}}{2} \quad : \quad \widehat{\mathcal{H}}_+ \rightarrow \mathcal{H}_-$$

$$\mathcal{D} = \begin{pmatrix} & D_{\text{SF}}^\dagger \\ D_{\text{SF}} & \end{pmatrix} \quad : \quad \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_- \rightarrow \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_-$$

$$\text{on } \Psi = \begin{pmatrix} \frac{1 - \hat{\Gamma}}{2} & \\ & \frac{1 + \Gamma}{2} \end{pmatrix} \psi_E \in \widehat{\mathcal{H}}_+ \oplus \mathcal{H}_-$$

- Eigenvalue

$$\lambda^2 = A_\mu^2 + B^2 + m^2, \quad A_\mu = M \frac{\sin p_\mu}{\sqrt{\lambda_W^2}}, \quad B = M \left( 1 + \frac{W}{\sqrt{\lambda_W^2}} \right)$$

$$\lambda_W^2 = \sin^2 p_\mu + W^2, \quad W = -M + \Sigma(1 - \cos p_\mu)$$

$$\tan p_0 T = -\frac{A_0}{m}$$

- In  $a \rightarrow 0$   $\lambda^2 \rightarrow p_0^2 + \vec{p}^2 + m^2, \quad \tan p_0 T = -\frac{p_0}{m}$

- Free propagator

$$\begin{aligned}
 G_{\text{SF}} &= 2 \frac{1 - \hat{\Gamma}}{2} \frac{1}{D} \frac{1 + \Gamma}{2} = 2 D^\dagger \frac{1 + \Gamma}{2} \frac{1}{DD^\dagger} \frac{1 + \Gamma}{2} \\
 &= D^\dagger (P_+ G_L + P_- G_R)
 \end{aligned}$$

$$G_R(x_0, y_0) = \frac{1}{2N_T} \sum_n \frac{1}{DD^\dagger(p)} \left( e^{ip_0(x_0 - y_0)} - e^{ip_0(x_0 + y_0)} \right)$$

$$G_L(x_0, y_0) = \frac{1}{2N_T} \sum_n \frac{1}{DD^\dagger(p)} \left( e^{ip_0(x_0 - y_0)} + e^{ip_0(x_0 + y_0)} \right)$$

$$DD^\dagger(p) = M^2 \frac{\sin^2 p_\mu}{\lambda_W^2} + M^2 \left( 1 - \frac{W}{\sqrt{\lambda_W^2}} \right)^2$$

$$p_0 = \frac{2n + 1}{2N_T} \pi, \quad n = -N_T + 1, \dots, N_T$$

- $a \rightarrow 0$  limit

$$G_{\text{SF}} \rightarrow G_{\text{SF}}^{(\text{cont.})}$$

- Phase of the determinant

- No  $\gamma_5$  Hermiticity ( $D_{\text{SF}}^\dagger \neq \gamma_5 D_{\text{SF}} \gamma_5$ )

$$D_{\text{SF}} = \frac{1 + \Gamma}{2} D \frac{1 - \hat{\Gamma}}{2}, \quad D_{\text{SF}}^\dagger = \frac{1 - \hat{\Gamma}}{2} D^\dagger \frac{1 + \Gamma}{2}$$

$$D_{\text{SF}}^\dagger = \gamma_5 u D_{\text{SF}} u^\dagger \gamma_5$$

- Hilbert space

$$D_{\text{SF}} : \widehat{\mathcal{H}}_+ \rightarrow \mathcal{H}_- \quad \Rightarrow \quad D_{\text{SF}} u^\dagger \gamma_5 : \mathcal{H}_- \rightarrow \mathcal{H}_-$$

$$u = \frac{1 + i\gamma_5 \Gamma}{2} (1 - aD) + \frac{1 - i\gamma_5 \Gamma}{2} \quad : \quad \text{local, unitary}$$

$$D_{\text{SF}} u^\dagger \gamma_5 = \frac{1 + \Gamma}{2} D u^\dagger \gamma_5 \frac{1 + \Gamma}{2}$$

- Determinant in  $\mathcal{H}_-$  subspace

$$\det_{\{\mathcal{H}_-\}} (D_{\text{SF}} u^\dagger \gamma_5) = \det \left( \frac{1 + \Gamma}{2} D u^\dagger \gamma_5 \frac{1 + \Gamma}{2} + \frac{1 - \Gamma}{2} \right)$$

- Phase of the determinant

$$D_{\text{SF}}^\dagger = \gamma_5 u D_{\text{SF}} u^\dagger \gamma_5 \Rightarrow (D_{\text{SF}} u^\dagger \gamma_5)^\dagger = (\gamma_5 u)^2 (D_{\text{SF}} u^\dagger \gamma_5)$$

$$e^{i\phi} = \sqrt{\det u} = \prod_{n \in \{+i\gamma_5 \Gamma\}} (1 - a\lambda_n)^{1/2}$$

$$u = \frac{1 + i\gamma_5 \Gamma}{2} (1 - aD) + \frac{1 - i\gamma_5 \Gamma}{2}$$

$$\begin{aligned} D\psi_n^{(+)} &= \lambda_n \psi_n^{(+)}, & (i\gamma_5 \Gamma)\psi_n^{(+)} &= +\psi_n^{(+)} \\ D\gamma_5 \psi_n^{(+)} &= \lambda_n^* \gamma_5 \psi_n^{(+)}, & (i\gamma_5 \Gamma)\gamma_5 \psi_n^{(+)} &= -\gamma_5 \psi_n^{(+)} \end{aligned}$$

(a) determinant is complex

(b) however phase is irrelevant  $O(a)$

localized at the boundary ( $\delta \ln \det u$ )

(c)  $N_f = 2 \Rightarrow$  divide the det by  $\det u \Rightarrow$  real & positive

## §4 Domain-wall fermion

$$S = \Sigma \bar{\psi} \left( \gamma_M D_M - \frac{1}{2} D^2 - M \right) \psi + m_f \bar{q} q$$

$$q = \frac{1 - \gamma_5}{2} \psi_1 + \frac{1 + \gamma_5}{2} \psi_{N_5}$$

- time reversal symmetry

$$\psi \rightarrow i\gamma_0\gamma_5 R P \psi, \quad \bar{\psi} \rightarrow \bar{\psi} i\gamma_0\gamma_5 R P, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- chiral symmetry (Furman-Shamir)

$$\psi \rightarrow -iQ\psi, \quad \bar{\psi} \rightarrow \bar{\psi} iQ, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- anti-periodicity  $\psi(x_0 + 2T) = -\psi(x_0)$

- **Orbifolding projection**

$$\begin{aligned}\psi(x) &= A\psi(x), & \bar{\psi}(x) &= \bar{\psi}(x)A, & A &= \gamma_0\gamma_5PQR \\ q(x) &= -\Gamma q(x), & \bar{q}(x) &= \bar{q}(x)\Gamma, & \Gamma &= \gamma_0R\end{aligned}$$

- **Orbifolded action**

$$S = \Sigma \bar{\psi} D_{\text{SF}} \psi, \quad D_{\text{SF}} = \frac{1+A}{2} D_{\text{dwf}} \frac{1+A}{2}$$

- $\gamma_5$  Hermiticity

$$D_{\text{SF}}^\dagger = \frac{1+A}{2} D_{\text{dwf}}^\dagger \frac{1+A}{2} = P\gamma_5 D_{\text{SF}} \gamma_5 P$$

- **Quark propagator**

$$\langle q\bar{q} \rangle_{\text{SF}} = \frac{1-\Gamma}{2} \langle q\bar{q} \rangle \frac{1+\Gamma}{2} \Rightarrow (N_5 \rightarrow \infty) \Rightarrow \frac{1-\Gamma}{2} \frac{2}{D_{\text{OV}}} \frac{1+\Gamma}{2}$$

- **Overlap Dirac fermion propagator**

$$G_{\text{SF}}^{\text{OD}} = \frac{1-\Gamma}{2} \frac{2}{D_{\text{OV}}} \frac{1+\Gamma}{2} + a \frac{1+\Gamma}{2} = 2 \frac{1-\hat{\Gamma}}{2} \frac{1}{D_{\text{OV}}} \frac{1+\Gamma}{2}$$

## §5 Conclusion

- A potential problem of zero mode with Dirichlet BC.
- Dirichlet BC in  $D_W$  does not work for OD and DWF.
- A criterion to introduce Dirichlet BC in  $D_{OV}$ ,  $D_{dwf}$ 
  - chiral symmetry (GW relation) breaking

$$(1 + \gamma_0)\psi|_0 = 0, \quad (1 - \gamma_0)\psi|_T = 0$$

- field theory with BC  $\Leftrightarrow$  orbifolded field theory
  - time reversal symmetry
  - chiral symmetry (massless theory)
  - anti-periodicity
- Orbifolded action

$$S = \Sigma \bar{\psi} D_{SF} \psi, \quad D_{SF} = \frac{1 + \Gamma}{2} \left( D + m\eta(x_0) \left( 1 - \frac{a}{2} D \right) \right) \frac{1 - \hat{\Gamma}}{2}$$

- SF Dirichlet BC
- Classical level, eigenvalue and propagator

- (technical) Problem

- (a) determinant is complex

- (b) however phase is irrelevant  $O(a)$

- localized at the boundary ( $\delta \ln \det u$ )

- (c)  $N_f = 2 \Rightarrow$  divide the det by  $\det u \Rightarrow$  real & positive